which a number belongs, indices, primitive roots, etc.) can in a very simple and satisfactory manner be demonstrated by means of a square table which for a given fixed prime modulus gives both the residues of a^λ for λ fixed, a variable; and for a fixed, λ variable.

Some apparently new results will be presented on another occasion.

10. Professor Jackson's paper discussed applications of the geometrical interpretation of correlation coefficients less simple than those which are treated in papers published in recent volumes of the Monthly. In particular, it gave a geometrical derivation of the regression coefficients for a problem involving three statistical variables.

The members and friends of the association were guests of the University of Denver at a banquet on the evening of April 11. President Gorrell acted as toastmaster. The address of welcome was given by Chancellor Frederick Hunter of the University of Denver. The response was given by Professor A. J. Kempner of the University of Colorado.

Following this a very interesting and instructive address was given by the guest of honor, Professor Dunham Jackson, on "The significance of elementary mathematics in modern statistics."

A. J. LEWIS, *Secretary*

THE SEVENTH ANNUAL MEETING OF THE INDIANA SECTION

The seventh annual meeting of the Indiana section of the Mathematical Association of America was held on May 2–3, 1930 at Earlham College, Richmond, Indiana.

There were forty-five present at the meeting including the following twenty-three members of the Association: W. C. Arnold, R. W. Babcock, Gladys L. Banes, G. E. Carscallen, P. T. Copp, C. S. Doan, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, C. T. Hazard, D. F. Heath, Cora B. Hennel, Florence Long, Juna M. Lutz, T. E. Mason, J. A. Reising, C. K. Robbins, L. S. Shively, K. P. Williams, W. A. Zehring.

On Friday at 5:30 P.M. a reception was given to the visiting members and their guests. At 6:30 P.M. a complimentary banquet which was held in the dining room of the college was attended by sixty guests of the college. Professor E. D. Grant presided at the banquet and introduced President Denny of Earlham College, who made a brief address of welcome. Music was provided during the banquet by a trio of students of the college.

At eight o'clock a short pipe organ recital was presented in Stoddard Auditorium. The public lecture of the evening, under the auspices of Earlham College, was given by Professor Louis C. Karpinski of the University of Michigan

on the subject, "Mathematics and the Eternal Verities." Professor Karpinski pointed out that work and intensive intellectual application are the foundations of success not only in mathematics but in all the arts and sciences upon which the progress of civilization rests. In mathematics the student is inevitably confronted with these realities more than in any other subject of the school curriculum. Particularly for students who are likely ever to do creative work mathematics furnishes stimulus and inspiration. Even also for the great mass of students in our secondary schools mathematics is one of the few studies in which the fact of intellectual work or lack of work is made evident in every recitation. When this fact is combined with the fact that mathematics is the indispensable foundation for the study of engineering, economics, physics, and many other sciences, it becomes evident that mathematics must continue to hold a prominent place in our secondary school program.

At ten o'clock Saturday morning in Carpenter Hall the following officers were elected for the coming year: Chairman, P. D. Edwards, Ball State Teachers College; Vice-Chairman, G. E. Carscallen, Wabash College; Secretary-Treasurer, H. T. Davis, Indiana University.

A chairman's address in absentia by Professor Zinszer on "Sub-atomic Versus Interstellar Space," was read by Professor Grant. The first part of the paper consisted of a rapid review of the discoveries in the field of modern physics and their mathematical agencies. This included a brief description of Millikan's determination of the electronic radius, also Rutherford's determination of the approximate size of the nucleus of the gold atom. Following an exposition on the method of parallax and its application to astronomy, the paper took up a discussion of the galactic system and its dimensions.

The following seven papers were read:

1. "On non-composite plane curves of the form $C_6{:}8A^2B^2$," by Professor J. C. Polley, Wabash College, by invitation.

2. "Uses of vectors in geometry and trigonometry," by Professor R. W. Babcock, DePauw University.

3. "Number one and number naught," by Dr. A. F. Bentley, Paoli, Indiana, by invitation.

4. "Amount of training in mathematics required of high school teachers of mathematics in the various states," by Professor P. D. Edwards, Ball State Teachers College.

5. "Technique of instruction for large classes in mathematics," by Mr. C. E. Trueblood, Arsenal Technical Schools, Indianapolis, Indiana, by invitation.

6. "The predicted location of the 1930 center of population of the United States," by Professor L. S. Shively, Ball State Teachers College.

7. "The minimum essentials' place in mathematics courses," by Professor G. H. Graves, Purdue University.

Abstracts of these papers follow:

1. In this paper the author considers the web of sextic curves of the form $C_6{:}8A^2$, i.e., with eight double points. It is shown that: (1) on any non-composite

sextic C_6 of the web there are six points where a member of the pencil of cubics $C_3 - kC_3^1 = 0$ and C_6 have a common tangent and each point is a ninth double point on a non-composite web; (2) the locus of the ninth double point is a curve of order nine of the form $C_9{:}8A^3$.

2. Professor Babcock thinks that the concept of vector is within the grasp of the average pupil of geometry. Various problems involving intersections of lines of projections of lines may easily be solved by vector algebra. Several of the formulae of elementary trigonometry are easily derived by means of the scalar product. This work may be used for special projects for students who are possessed of intellectual curiosity.

3. Under Hilbertian technique, though not Hilbertian minimal presupposition, we may define a pure mathematics as any system of full consistency. Examining inductive numbers with the approach of Kronecker or Poincaré, rather than with the scaffoldings of Russell or of the Mengenlehre, we obtain a pure theory of number to which the distinctions of cardinal and ordinal are irrelavent. The cardinal is relegated to the "impure" or "foundation" regions of mathematics. Realistic and analytic (semantic) postulates for the investigation of Number One are constructed, and analysis is carried through by differentiation of operative zeros from realistic nulls, and by similar differentiation of infinities. A semantic number series and a semantic radix series are constructed. Under this construction proof is given that decimals are denumerable. The Cantorian proof of non-denumerability yields under analysis its realistic elements and fixations. Analysis of the ordinary form of proof that Null is a Zahl yields similar evidence of confusion between semantic symbol and realistic reference.

4. Practically all the states issue from two to a dozen different grades of high school certificates. The author discussed only the requirements for the certificates of highest grade. In most of the states a certificate of general validity for all subjects is granted. Consequently the teacher of high school mathematics may have studied no mathematics of college grade. In addition to a discussion of these requirements for the various states the author suggested some action on the part of college instructors of the state might improve the secondary teaching in Indiana.

5. During the past five or six years Mr. Trueblood has been experimenting with classes of a hundred students and finds that the results are satisfactory both from the standpoint of the teacher and the students. In this paper he outlined the method of conducting large classes and pointed out the special technique necessary.

6. Professor Shively predicted the location of the center of population of the United States basing his calculation upon estimates of population increase and of the distribution of population during the preceding decade. The results of the calculation, which was made as of January 1, 1930, are that the center has moved westward 13.2 miles and northward 2.8 miles with probable errors

of 1.25 and .84 miles respectively. This places it in N. Lat. 39°13′ and W. Long. 86°58′, a point a little to the southwest of Arney in Owen County, Indiana.

7. In this paper the author sketched a plan in use at Purdue University. The course is divided into a few heads and after the class periods allotted to each of these have been devoted to discussion, recitation, and illustrative examples as usual, a test is given which determines, with the class work, whether the student has "cleared his record" on that head. No grades are given during the course and only those who clear their record under all the heads by the end of the term receive credit for the course.

At the close of the meeting a resolution was adopted by the members expressing their appreciation to Earlham College and to the mathematics department for the splendid banquet and their efforts in making the meeting a success. Also the section expressed its appreciation to Professor Karpinski and Dr. Bentley.

V. V. LATSHAW, *Acting Secretary*

THE FOURTEENTH MEETING OF THE KENTUCKY SECTION

The fourteenth regular meeting of the Kentucky Section of the Mathematical Association of America was held at Transylvania College, Lexington, Kentucky, on Saturday, April 15, 1930. The Section was fortunate in having as its guest Professor W. D. Cairns, Secretary-Treasurer of the Association.

There were forty-three present, including the following twenty-one members of the Association: P. P. Boyd, W. D. Cairns, C. E. Caldwell, M. G. Carman, M. C. Dame, J. M. Davis, D. S. Dearman, A. R. Fehn, W. W. Garnett, Charles Hatfield, W. R. Hutcherson, C. G. Latimer, Elizabeth LeStourgeon, Mrs. A. R. Lyon, C. A. Maney, W. L. Moore, Smith Park, Sallie Pence, D. W. Pugsley, J. H. Simester, Guy Stevenson.

The chairman, Professor C. A. Maney, presided at both the morning and afternoon sessions. All present were guests of Transylvania College at luncheon.

The officers elected for the coming year were: Chairman, Professor J. M. Davis, University of Kentucky; Secretary, Professor A. R. Fehn, Centre College.

The program of the meeting was as follows:

Morning Session, 10:00 A.M.

1. "A certain identity in theta functions" by Mr. Smith Park, University of Kentucky and Eastern State Teachers' College.

2. "Trigonometric formulae by vector analysis" by Professor W. R. Hutcherson, Berea College.

3. "Mathematics—What's the use?" by Professor J. M. Davis, University of Kentucky.

4. "Difference equations" by Professor M. G. Carman, Murray State Teachers' College.

5. "A graphical solution of the equation $x^n+ax^2+bx+c=0$" by Professor Walter L. Moore, University of Louisville.

6. "Some simple methods and problems" by Professor J. M. Maxey, Asbury College, by invitation.

7. "Queen Dido's problem" by Professor Elizabeth LeStourgeon, University of Kentucky.

Luncheon 1:00 P.M.

8. "Current mathematical activities" by Professor W. D. Cairns, Oberlin College.

Afternoon Session, 2:30 P.M.

9. Mathematics for students of chemistry" by Professor A. R. Fehn, Centre College.

10. "The fundamental mathematical requirements of biology" by Professor D. S. Dearman, Kentucky Wesleyan College.

11. "The lure of mathematics" by Professor W. D. Cairns, Oberlin College.

12. Business Meeting.

Brief abstracts of some of these papers follow:

1. In this paper Mr. Park considered two sets S and S_1 of elements of algebraic numbers having a one-to-one correspondence. Two functions containing theta functions were constructed, the terms of the first being in one-to-one correspondence with the elements of S and the second with the elements of S_1. The terms of the two functions were thus in one-to-one correspondence, and by properly defining the variables of the first as linear functions of the variables of the second, making corresponding terms equal, he obtained the identity.

2. By use of two unit vectors A and B which make angles a and b with the i-axis, the dot and cross products produced the formulae for $\cos(b-a)$ and $\sin(b-a)$. The fundamental law for oblique spherical triangles, $\cos b=\cos c\cos a+\sin c\sin a\cos B$, was obtained by expansion of the expression $[A\times B]\cdot[C\times D]$, setting vector $B\equiv$vector D.

3. This paper discussed the importance of teachers of mathematics being prepared to answer this question whenever and however asked, and in the answer to seek to bring the questioner to realize that, aside from its all-pervading influence in all arts and science—indeed in all life, mathematics has a beauty, a harmony, a charm all its own; that it is in fact the most perfect means man has devised for discovering and realizing truth.

5. The nomographic method of D'Ocagne for the graphical solution of the equation $x^n+ax+b=0$ is extended to apply to the equation $x^n+ax^2+bx+c=0$ by the use of the parabola as the variable curve instead of a straight line. A mechanism based on Pascal's theorem is used to trace the arcs of the parabola.

7. This paper was a brief exposition of the method of determining first necessary conditions for a minimum or maximum in the case of the isoperimetric problem of the calculus of variations when the end points move along a fixed curve, with application to "Queen Dido's Problem," as related by Virgil.

9. This was a discussion of a paper by Professor F. Daniels of the University of Wisconsin reported in the January, 1928 issue of the American Mathematical Monthly.

10. This paper was a review of the article entitled "The fundamental mathematical requirements of biology" by J. Arthur Harris that appeared in the April, 1929, issue of the American Mathematical Monthly.

ELIZABETH LESTOURGEON, *Secretary*

UNIQUE DECOMPOSITION

By E. T. BELL, California Institute of Technology

I. AN ARITHMETICAL PROBLEM

1. *Introduction.* Before proceeding to a precise formulation of the problem of unique decomposition, it will be well for clearness to give a general description of its nature and of the situation in which it arises. What follows is merely descriptive; the exact statement can be given only after the postulational definitions of certain algebraic varieties—ova, commutative semigroups, modules, rays, rings, fields, irregular fields—in Section II. A fair image of the exact problem can be seen by having in mind as a background either the multiplicative or the additive part, but not both, of ordinary rational arithmetic, or the theory of the ideals only of an algebraic number field, the integers of the field being replaced by their corresponding principal ideals, or the set of all symmetric polynomials in n indeterminates ($n>1$), with rational integer coefficients, and their unique representations as polynomials in the relevant elementary symmetric functions. The brief descriptions of the algebraic varieties concerned are strictly provisional until the exact postulational statements.

2. *Algebraic Varieties.* A field is a set of elements closed under the four rational operations, division by zero excluded, and having unique zero and unit elements. A product is equal to zero only if one of its factors is zero. This is in distinction to the next.

An irregular field differs from a field only in the exclusion of each of m elements, $m>1$, as a divisor; m may be finite or infinity. The excluded elements are called irregular; the rest, regular. With respect to addition, multiplication, subtraction, irregular elements are indistinguishable from regular. The product of two irregular elements, neither the zero element, may be equal to the zero element, in which case the factors are called divisors of zero.

A set of elements closed under addition, multiplication and subtraction is called a ring.

A set of elements closed under addition and subtraction is called a module.

A set of elements closed under multiplication and division, the divisor being regular, is called a ray.

It may be remarked that the definitions of rings, modules and rays in the literature lack uniformity. For example, a module is sometimes defined as a set closed under addition, or under subtraction, without the explicit statement of the postulate that these operations shall have unique inverses. The sets closed under addition alone or multiplication alone, when only the associative and commutative laws are assumed, are identical abstractly, and they appear to be more fundamental for arithmetic than any of the foregoing. From such sets, we construct the others.

A set closed under a single operation R obeying the associative and commutative laws, and further subject to the postulate that $xRy = xRz$ implies $y = z$, is called a commutative semigroup. If a commutative semigroup contains only a finite number of distinct elements, it is an abelian (commutative) group. It is to be noted particularly that a commutative semigroup does not necessarily contain a modulus (element having with respect to R and elements of the set the properties of an identity); if the set does have a modulus, the modulus is necessarily unique. Inverses do not exist in a commutative semigroup without restriction by the adjunction of additional postulates. If such adjunction be made, we descend to modules or other less general varieties.

If from the definition of a commutative semigroup the postulate of cancellation, $xRy = xRz$ implies $y = z$, be deleted (as there exist sets for which it is false, for example in the algebra of logic), we obtain a still more rudimentary variety which does not seem to have been named. All that follows can be constructed from this variety. Lacking a better name, it may be called an ovum.

A variety which contains only either a finite number or a denumerably infinite number of distinct elements will be said to be countable. The elements of a countable variety can be ordered in 1, 1 correspondence with the positive rational integers if the variety is infinite, or with the first n integers if the variety contains precisely n (n finite) distinct elements. When such a correspondence is established, we shall say that the variety is ordered.

3. *The Problem.* Call the operation R under which the ovum Ω is closed composition, and the element of Ω obtained by composition from two or more elements of Ω their composite. An example of Ω occurs in the classical theory of composition of binary quadratic (arithmetical) forms.

Suppose now that Ω contains a countable set I of elements, and further that I contains a set P of elements such that:

3.1 Any element of I not in P is, except for permutations of the elements in the composite, uniquely a composite of elements in P.

3.2 No element in P is a composite of elements in P.

3.3 The sets I, P are maximal sets in Ω having the properties (3.1), (3.2). Then the elements of I may be called (by an obvious isomorphism) the rational integers of Ω, and those of P the rational primes; (3.1) – (3.3) express the law

of unique decomposition of integers in Ω into primes. More than one maximal set of integers or of primes in a set of integers may exist. Examples have been indicated in what precedes in which composition is multiplication, or addition, both being interpreted as in rational arithmetic, and in which it is neither. In view of this, we shall call 3.1 – 3.3 and their implications the composite arithmetic Λ of Ω. It is evident that so long as we remain in Ω, no further development of its arithmetic—properties of its rational integers—is possible. Thus, for example, if composition be identified with the associative and commutative properties of either multiplication or addition, it is identified with those of both. This leads to the following problem, of which we shall present a denumerable infinity of solutions, as the process of construction of one solution is iterable. The complete arithmetic of the ordinary rational integers is presupposed.

Given the composite arithmetic Λ of Ω to obtain from it the following:

3.4 An ordering of its rational integers.

3.5 A set of functions (to be called elements of V) of integers of such that, with respect to suitably defined rational operations the elements of V form an irregular field V. If at any stage of the new processes necessary, elements undefined in the data are introduced, the new elements are to be adjoined to the data in such a manner that the enlarged set is self-consistent. The definition of the new processes will always be such that consistency is automatically provided. The adjoined elements, of course, demand in the interpretations of the original data extensions of given properties. Such extensions can be given only with reference to the specific nature of the original data. For example, the adjunction necessary when the original elements are integral ideals, is that of fractional ideals.

This is the first, or algebraic part of the problem. The second part is arithmetical. Three cases arise, according as the number of distinct elements in V is finite, or denumerably infinite, or neither. Many instances of the first and second exist, and at least one of the third. However, we shall attend only to the first and second, and particularly to the second, as the necessary modifications in what follows when V is finite are obvious, and the second is the case of greater interest. We shall assume then that the elements of V are denumerably infinite. The second part of the problem follows.

3.6 To select from V a countably infinite ring R of its elements such that, with a suitable definition of arithmetical divisibility for irregular elements, R has a complete arithmetic A, additive and multiplicative, isomorphic with that of the ordinary rational integers, order relations included.

3.7 To superimpose V, A upon Ω, Λ so that every proposition concerning V (or A) has a unique correspondent in Ω (or Λ) which is true or false according as its correspondent in V (or A) is true or false.

Finally, with the necessary restrictions imposed by irregularity, we may proceed to

3.8 Construct continua from A abstractly identical with those of the real or complex number systems.

A solution of 3.6 will automatically produce an ovum from which the whole abstract process may be generated afresh, and so on indefinitely.

The problem of constructing a complete arithmetic from the composite arithmetic alone of an ovum is here solved through an irregular field. I have also given a solution (or rather an infinity of solutions), in another paper, through a field. The solution by an irregular field is, however, intrinsically richer, as it necessarily demands a definition of arithmetical divisibility for divisors of zero and a unique decomposition of such divisors into primes. This, I believe, has not hitherto been done for divisors of zero in the arithmetic of any algebra. The reader who prefers may pass at once to IV, using II, III only for reference.

II. ALGEBRAIC VARIETIES

4. *Variety; Notation.* As each of the four rational operations in an irregular field is ultimately to receive an infinity of distinct interpretations, the customary $+, \cdot, -, \div$ will be replaced in the postulates by $*, \dagger, \S, \ddagger$, and each of these symbols has only the meaning assigned to it in the postulates of the variety concerned. For example, in an ovum, it is meaningless to say that $*$ is algebraic addition, as no inverse to $*$ is defined in the ovum. But, inversely, a true proposition in an ovum remains true in a module, a ring, a commutative semigroup, a field, etc., which contains the ovum.

4.1 *Definition.* A *variety* V is a system consisting of a relation $\sim$, called *equivalence*, a set Σ of marks $\alpha, \beta, \cdots, \gamma, \cdots$, called *elements* of Σ or of V, and one or more *operations* $*, \dagger, \S, \ddagger, \cdots$, which can be performed upon any pair α, β of elements of V to produce uniquely determined elements $\alpha * \beta, \alpha \dagger \beta, \alpha \S \beta, \alpha \ddagger \beta, \cdots$, of V such that the postulates 4.11–4.15, 4.17 for $\sim, *, \S, \ddagger, \cdots$, are satisfied.

4.11 If α, β are any elements of V, $\alpha \sim \beta$ is significant (well defined, and either true or false).

4.12 $\alpha \sim \alpha$.

4.13 If $\alpha \sim \beta$, then $\beta \sim \alpha$.

4.14 If $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

4.15 If $R(\alpha, \beta, \cdots, \gamma, \cdots)$, is any relation between elements $\alpha, \beta, \cdots, \gamma, \cdots$, of V, and $\alpha \sim \alpha', \beta \sim \beta', \cdots, \gamma \sim \gamma', \cdots$, then $R(\alpha, \beta, \cdots, \gamma, \cdots)$, $R(\alpha', \beta', \cdots, \gamma', \cdots)$, are both true or both false.

4.16 *Definition.* If $\alpha \sim \beta$ is false, we write $\alpha \sim' \beta$.

4.17 If $\dagger\dagger$ and $\S\S$ are members of the set $*, \dagger, \S, \ddagger, \cdots$, and are not the same there exists in V at least one pair of elements α, β for which $\alpha \dagger\dagger \beta \sim' \alpha \S\S \beta$.

It is evident that equivalence, $\sim$, has with respect to elements of V all the properties of equality, $=$, with respect to real numbers, being reflexive (4.12), symmetrical (4.13), and transitive (4.14), while 4.15 permits the substitution of equivalent elements. Nevertheless, in the instances adduced later, $\sim$ bears but slight resemblance to $=$. For example, if (α, β, γ) is a one-rowed matrix, and $(\alpha, \beta, \gamma) = (\alpha', \beta', \gamma')$, the usual definition gives $\alpha = \alpha', \beta = \beta', \gamma = \gamma'$, and it does not apply to $(\alpha, \beta, \gamma) = (\alpha', \beta')$, whereas $\sim$ does.

4.18 *Definition.* If to the preceding postulates others be added, the resulting system will be called a *special variety*, provided the entire set of postulates is consistent.

Consistency can always be proved for a special variety by exhibiting an instance which satisfies the postulates. In what follows, all consistency proofs are deferred to IV, where all are implied simultaneously by the construction of the systems constructed.

5. *Commutative, Associative and Distributive Special Varieties.*

5.1 *Definition.* A special variety V is said to be *commutative* with respect to the operation $*$ of V if the postulate 5.11 is satisfied.

5.11. If α, β are any elements of V, then $\alpha * \beta \sim \beta * \alpha$.

5.2 *Definition.* A special variety V is said to be *associative* with respect to the operation $*$ of V if the postulate 5.21 is satisfied.

5.21 If α, β, γ are any elements of V, then $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.

5.3 *Definition.* A special variety V is said to be *left-distributive* with respect to the ordered pair $(\dagger, *)$, where $\dagger, *$ are operations of V and are not the same, if the postulate 5.31 is satisfied.

5.31 If α, β, γ are any elements of V, then

$$\alpha \dagger (\beta * \gamma) \sim (\alpha \dagger \beta) * (\alpha \dagger \gamma).$$

5.4 *Definition.* If instead of 5.31 the postulate 5.41 is satisfied, V is said to be *right-distributive* with respect to the ordered pair $(\dagger, *)$.

(5.41) If α, β, γ are any elements of V, then

$$(\alpha * \beta) \dagger \gamma \sim (\alpha \dagger \gamma) * (\beta \dagger \gamma).$$

5.5 *Definition.* If the special variety V is both *left-distributive* and *right-distributive* with respect to the ordered pair $(\dagger, *)$, it is said to be *distributive* with respect to $(\dagger, *)$.

It is unnecessary here to develop the numerous consequences of these postulates, as only the simplest of them will be required.

6. *Modular Special Varieties.* A modulus is an identity element with respect to a given operation.

6.1 *Definition.* If in the special variety V there exists an element μ_* such that the postulate 6.11 is satisfied, V is said to be *modular* with respect to $*$.

6.11 If α is any element of V, then at least one of the following holds,

$$\alpha * \mu_* \sim \alpha \qquad \mu_* * \alpha \sim \alpha.$$

6.2 *Definition.* If only $\alpha * \mu_* \sim \alpha$ in 6.11 holds, μ_* is called a *right-modulus*; if only $\mu_* * \alpha \sim \alpha$ holds, μ_* is called a *left-modulus*; if both equivalences hold, μ_* is called a *modulus* with respect to $*$.

7. *Regular and Irregular Special Varieties.*

7.1 *Definition.* If V is a modular special variety having with respect to $*$ the modulus $\mu*$, and if α, β are elements of V such that $\alpha * \beta \sim \mu*$, α is called a *left-*

inverse of β with respect to $*$, and β a *right-inverse* of α with respect to $*$, and each of α, β is said simply to be an *inverse* of the other with respect to $*$.

7.2 *Definition.* A modular special variety is said to be right-regular if the postulate 7.21 holds, left-regular with respect to $*$ if the postulate 7.22 holds.

7.21 If α is any element of V, α has with respect to $*$ a unique right-inverse.

7.22 If α is any element of V, α has with respect to $*$ a unique left-inverse.

7.3 *Definition.* A modular special variety V is said to be *regular* with respect to $*$ if it is both right-regular with respect to $*$ and left-regular with respect to $*$.

7.4 *Definitions.* An element of a special modular variety V which does not have a unique right (left) inverse with respect to the operation $*$ of V, is said to be *right- (left) irregular* with respect to $*$; an element of V which is both right-irregular and left-irregular with respect to $*$ is said to be *irregular* with respect to $*$.

If V is right-irregular (or left-irregular, or irregular) and contains precisely m (m necessarily >0) right-irregular (or left-irregular, or irregular) elements with respect to $*$, we say that the *index of right-irregularity* (or *of left-irregularity* or *of irregularity*) of V with respect to $*$ is m.

As the definitions are similar for all three possibilities, we may state only those for irregularity.

If every element of V is irregular with respect to $*$, V is said to be *totally irregular* with respect to $*$. It is perhaps not obvious that totally irregular commutative and associative varieties exist; an infinity of such are given by the irregular elements, other than the unique zero element, of the matric varieties constructed in IV.

By referring to 7.4 and 7.2 we see that an element may be irregular in only one of two distinct respects: the element may have no inverse; the element may have more than one inverse. In the totally irregular varieties mentioned above each element has an infinity of inverses.

8. *Ordered Varieties.* The postulates for order fall into several cases, according to the degree of specialization of the variety concerned, each more restrictive than its predecessor. The most restricted set, which applies to rings and fields, regular or irregular, is abstractly identical with the postulates for inequality in the real number system.

8.1 *Definition.* If a variety V or a special variety V is *ordered* with respect to the relation $\rightarrow$, it is necessary that the postulates 8.11, 8.12 hold.

8.11 If α, β are any elements of V, one and only one of the following relations is true,

$$\alpha \sim \beta, \quad \alpha \rightarrow \beta, \quad \beta \rightarrow \alpha.$$

8.12 If α, β, γ are any elements of V such that $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$, then $\alpha \rightarrow \gamma$.

8.2 *Definition.* The relation $\alpha \leftarrow \beta$ means $\beta \rightarrow \alpha$.

The preceding definitions apply to all varieties, and in particular to the V in 4.1, which is unspecialized. Specialization enters through the imposition of further postulates concerning an operation $*$ of V. *If V is unspecialized*, we say

that 8.11, 8.12 *are the necessary and sufficient conditions that V be ordered with respect to* $\rightarrow$.

There is an extensive theory possible for ordered special varieties. Of this, for brevity, we shall state only the postulates for ordered modular varieties, as these are the minimum necessary for the subsequent theory of unique decomposition. For the necessary definitions, see Section 6. Again for simplicity we shall attend only to the case in which the special variety has a modulus (as defined in 6.2).

8.3 *Definition.* The modular special variety V having with respect to $*$ the modulus μ_* is said to be *ordered* with respect to the relation $\rightarrow$ when and only when, in addition to 8.11, 8.12, the postulate 8.31 is satisfied.

8.31 If α, β are any elements of V such that $\alpha \rightarrow \beta$, and if γ is any element of V, such that $\gamma \rightarrow \mu_*$, then $\alpha * \gamma \rightarrow \beta * \gamma$.

To proceed, we refer to 5.1, 5.2, 5.5, 6.2, 7.2, 7.3.

8.4 *Definition.* A special variety V which is

8.41 commutative and associative with respect to each of the distinct operations $*$, $\dagger$;

8.42 distributive with respect to the ordered pair $(\dagger, *)$;

8.43 regular with respect to $*$ and having the modulus μ_*, is said to be *ordered* with respect to $\rightarrow$ if and only if the postulates 8.44, 8.45 are satisfied in addition to 8.11, 8.12.

8.44 If α, β are any elements of V such that $\alpha \rightarrow \beta$, and γ is any element of V, then $\alpha * \gamma \rightarrow \beta * \gamma$.

8.45 If α, β, γ are any elements of V such that $\alpha \rightarrow \beta$ and $\gamma \rightarrow \mu_*$, then $\alpha \dagger \gamma \rightarrow \beta \dagger \gamma$.

Part of this is implied by previous definitions and postulates, but on account of its importance for the sequel it has been stated in full.

III. CERTAIN SPECIAL VARIETIES

9. *Named Varieties.* We shall drop the designation "special," and speak, for example, simply of a commutative variety instead of a commutative special variety.

9.1 *Ovum.* A variety defined by 4.1, 5.1, 5.2 is called an *ovum* with respect to $*$.

An ovum is hence a variety commutative and associative with respect to a single operation. Although an ovum is completely defined by 9.1 it may be well for clearness to state some of the properties which can *not* be assumed in it. These are, chiefly, the existence of a modulus, the existence of inverses with respect to $*$, and the cancellation postulate as stated in 9.22. Even when an ovum is contained in a variety which does have one of these properties, that property can not be assumed in the ovum itself. Similar remarks apply to all special varieties.

9.2 *Commutative semigroup.* An ovum V with respect to $*$ which satisfies the postulates 9.21–9.23 is called a commutative semi-group with respect to Σ, $*$.

9.21 Every element of Σ is in V.

9.22 If σ is any element of Σ, and α, β are any elements of V, such that $\alpha * \sigma \sim \beta * \sigma$, then $\alpha \sim \beta$.

9.23 Σ is not null and is the maximal[1] set for which 9.22 is satisfied.

9.24 *Definition.* If Σ in 9.2 coincides with V (that is, if every element of Σ is in V, and every element of V is in Σ), V is called a *commutative semigroup* with respect to $*$.

9.25 *Theorem.* If a commutative semigroup contains a modulus, the modulus is unique.

This is either well known or proved at once from the definitions.

9.3 *Module.* Here we shall define only the simplest modules, in which the composition $*$ is commutative. For this we refer to 7.3, 7.4.

9.31 *Regular Module.* A commutative modular variety which is regular with respect to $*$ is called a *module* with respect to $*$.

9.31 *Irregular Module.* A commutative modular variety which is irregular with respect to $*$, the index of irregularity being $m(\geqq 1)$, is called an *irregular module* of index m.

9.4 *Ray.* This introduces nothing new, and is included in order to take account of the term as it has been used in the literature.

9.41 *Definition.* An irregular module of index 1 is called a ray.

9.5 *Ring.* A variety satisfying the postulates 8.41, 8.42, 8.43 is called a *ring* with respect to $(\dagger, *)$.

A simple example of a ring is the set of all rational integers with respect to $(\times, +)$; the modulus is 0. In the definition it is to be noted particularly that the ring is not postulated to be modular with respect to $\dagger$.

9.6 *Field.* A variety V is called a *field* with respect to $(\dagger, *)$ if the postulates 9.61–9.64 are satisfied.

9.61 V is a ring with respect to $(\dagger, *)$.

9.62 V is a ray with respect to $\dagger$.

9.63 The irregular element of the ray in 9.62 is μ_*, the modulus of the ring in 9.61.

9.64 μ_* is distinct from the modulus $\mu_\dagger$ of the ray in 9.62.

It is readily seen that 9.6 accords with any of the usual complete definitions of a field, when $(\dagger, *) \equiv (\cdot, +)$, $(\mu_\dagger, \mu_*) \equiv (1, 0)$, 1, 0 being the unit and zero elements of the field.

9.7 *Irregular Field.* A variety V is called an *irregular field* with respect to $(\dagger, *)$ if the postulates 9.71–9.74 are satisfied.

9.71 V is a ring with respect to $(\dagger, *)$.

9.72 V is an irregular module with respect to $\dagger$, with index of irregularity $m > 1$.

9.73 μ_* is an irregular element of the module in 9.62.

9.74 μ_*, $\mu_\dagger$ are distinct.

[1] A set S is said to be *maximal* with respect to a property P if every element of S has the property P, and S is not contained in a set having elements not in S which have the property P.

9.8 *Definition.* The *index* of the irregular field in 9.7 is m. An irregular field of index 1 is a field (by definition).

IV. CERTAIN MATRIC VARIETIES

10. *Notation.* For ease in following the isomorphisms between varieties whose elements are real or complex numbers and those considered in this section, also to prepare for an extensive generalization indicated later, we shall use for matrices the signs in the second row following, for real or complex numbers those in third, the signs in any column of the table being instances of the first sign in that column, whose properties have been defined in preceding sections. The meanings of §, ‡, which were not explicitly used, are sufficiently clear as the operations inverse to *, † respectively when such inverses exist in the system to which *, † refer. The table is (n is an integer >0),

$$\begin{array}{cccccccccc} * & , & \dagger & , & \S & , & \ddagger & , & \sim & , & \sim' & , & \rightarrow & , & \leftarrow ; \\ (+)_n, & & (\cdot)_n, & & (-)_n, & & (\div)_n, & & (=)_n, & & (\neq)_n, & & (>)_n, & & (<)_n; \\ + & , & \cdot & , & - & , & \div & , & = & , & \neq & , & > & , & < . \end{array}$$

In the last row we shall use ab and a/b as customary for $a \cdot b$, $a \div b$ when convenient. The significance of the second row with n omitted is explained in Section 15.

11. *One-rowed Matrices.* A *real, finite one-rowed matrix* of *order* n (n finite) is an ordered set $(x_1, x_2, \cdots, x_n)$ of n real finite numbers $x_1, x_2, \cdots, x_n$. We shall call $(x_1, \cdots, x_n)$ an *M_n-number of order n*, or briefly M_n-number. The *coordinates* of $(x_1, \cdots, x_n)$ are the numbers $x_1, \cdots, x_n$ in this order.

11.1 *Equivalence of M_n-numbers.* We shall say that the M_n-numbers $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ are *equivalent*, and write

$$(x_1, \cdots, x_n)(=)_n(y_1, \cdots, y_n)$$

when and only when $x_j = y_j (j = 1, \cdots, n)$.

11.12 *Theorem.* Equivalence of M_n-numbers is an instance of abstract equivalence, $\sim$, in Section 4, up to and including 4.12.

The postulate 4.14 for $\sim$ will be seen to hold as we proceed. It is evident that the whole of the second line of the table of Section 10 must first be defined for M_n-numbers.

11.2 *Definition.* An M_n-number is said to be *regular* or *irregular* according as none or at least one of its coordinates is zero.

11.3 *Definition.* The M_n-number ζ_n each of whose coordinates is zero, is called the *zero M_n-number.*

11.4 *Definition.* The M_n-number η_n each of whose coordinates is 1 is called the *unit M_n-number.*

12. *Rational Operations upon M_n-Numbers.* Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ be any M_n-numbers, and $(z_1, \cdots, z_n)$ any regular M_n-number. The rational operations upon M_n-numbers are defined by (12.1)–(12.4).

12.1 $(x_1, \cdots, x_n)(+)_n(y_1, \cdots, y_n)(=)_n(x_1 + y_1, \cdots, x_n + y_n)$.

12.2 $(x_1, \cdots, x_n)(-)_n(y_1, \cdots, y_n)(=)_n(x_1 - y_1, \cdots, x_n - y_n)$.

12.3 $(x_1, \cdots, x_n)(\cdot)_n(y_1, \cdots, y_n)(=)_n(x_1y_1, \cdots, x_ny_n)$.

12.4 $(x_1, \cdots, x_n)(\div)_n(z_1, \cdots, z_n)(=)_n(x_1/z_1, \cdots, x_n/z_n)$.

The following are obvious from the definitions.

12.5 *Theorem.* With respect to each of $(+)_n$, $(\cdot)_n$, the set of all M_n-numbers is an ovum (section 9).

12.6 *Theorem.* With respect to $(+)_n$ the set of all M_n-numbers is a commutative semigroup (9.24), and a module (9.31) in which the element inverse to any element $(x_1, \cdots, x_n)$ is $(-x_1, \cdots, -x_n)$ and the modulus is ζ_n.

12.7 *Theorem.* With respect to $(\cdot)_n$ the set of all regular M_n-numbers is a commutative semigroup (9.24).

12.8 *Theorem.* The set of all M_n-numbers is a ring (9.5) with respect to $((\cdot)_n, (+)_n)$, and an irregular field of index 1 or ∞ according as $n=1$ or $n>1$, the irregular elements being the set of all irregular M_n-numbers (9.7, 9.8).

The case $n=1$ is merely that of the *field* of all real numbers.

13. *Ordered M_n-Numbers.* We refer for the notation to Section 10, and for the definitions and postulates concerning order to Section 8.

13.1 *Definition.* We say that the M_n-number $(x_1, \cdots, x_n)$ has the relation $(>)_n$ to the M-number $(y_1, \cdots, y_n)$, when and only when one, and necessarily only one, of the following n sets of conditions $G_j(x, y)(j=1, \cdots, n)$ is satisfied:

$$G_1(x, y) \equiv x_1 > y_1;$$

$$G_2(x, y) \equiv x_1 = y_1,\ x_2 > y_2;$$

. . . . . .

$$G_{n-1}(x, y) \equiv x_1 = y_1,\ x_2 = y_2, \cdots, x_{n-2} = y_{n-2},\ x_{n-1} > y_{n-1};$$

$$G_n(x, y) \equiv x_1 = y_1,\ x_2 = y_2, \cdots, x_{n-1} = y_{n-1},\ x_n > y_n.$$

When one of these sets is satisfied, we write

$$(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n).$$

13.2 *Definition.* Denote by $L_j(x, y)$ the result of replacing $>$ by $<$ in $G_j(x, y)$. Then we say that the M_n-number $(x_1, \cdots, x_n)$ has the relation $(<)_n$ to the M_n-number $(y_1, \cdots, y_n)$, and write

$$(x_1, \cdots, x_n)(<)_n(y_1, \cdots, y_n),$$

when and only when one (and necessarily only one) of the sets of conditions $L_j(x, y)(j=1, \cdots, n)$ is satisfied.

13.3. *Theorem.* The ring of all M_n-numbers in 12.8 is ordered with respect to $(>)_n$, except for the property 8.45 (exception removed, §19, end).

As the proofs necessary for 13.3 are all similar, we shall give that for only one part, and we shall show that if $(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n)$, and

$(y_1, \cdots, y_n)(>)_n(z_1, \cdots, z_n)$, where $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, $(z_1, \cdots, z_n)$ are any M_n-numbers, then $(x_1, \cdots, x_n)(>)_n(z_1, \cdots, z_n)$. The hypotheses imply the existence of integers i, j, $1 \leqq i, j \leqq n$, such that $G_i(x, y)$ and $G_j(y, z)$ both hold. If $i=j$, everything is proved. If $i \neq j$, let $[i, j]$ denote the smaller of i, j. Then the hypotheses imply $G_{[i,j]}(x, z)$ and the conclusion follows.

Notice that $(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n)$ and $(y_1, \cdots, y_n)(<)_n(x_1, \cdots, x_n)$ are indeed, as they should be, identical statements.

14. *Least Equivalent Sets.* The order of an M_n-number is n (defined in section 11, beginning). We have now to consider M_n-numbers of the orders $n=1, 2, 3, \cdots$, simultaneously. The process may be briefly described as follows. Given any set of such numbers of different orders: the members of the set are replaced by other numbers, all of the same order, and "equivalence" is so defined for the two sets that any relation in either has a unique equivalent correspondent in the other. Operations upon numbers of different orders are thus referred to the corresponding operations upon numbers of the same order. We proceed to make this precise.

Let Ω_j denote a sequence of precisely j terms each of which is zero, and define Ω_0 by $(x_1, \cdots, x_n, \Omega_0) \equiv (x_1, \cdots, x_n)$, where $(x_1, \cdots, x_n)$ is an arbitrary real matrix of any finite order n. Let $X', Y', \cdots, Z'$ be k real matrices, k finite, of the respective orders $n_1, n_2, \cdots, n_k$, and let n be the least integer $\geqq 0$ such that

$$n \geqq n_1, \ n \geqq n_2, \cdots, n \geqq n_k.$$

The *conjoint* (U, V) of $U \equiv (u_1, \cdots, u_a)$, $V \equiv (v_1, \cdots, v_b)$, in this order, is $(u_1, \cdots, u_a, v_1, \cdots, v_b)$.

Define now $X, Y, \cdots, Z$ as the respective conjoints

$$X \equiv (X', \Omega_{n-n_1}), \ Y \equiv (Y', \Omega_{n-n_2}), \cdots, Z \equiv (Z', \Omega_{n-n_k}).$$

Then $X, Y, \cdots, Z$ are all of the same order n, and are said to be *equivalent* respectively to $X', Y', \cdots, Z'$; further, the set $X, Y, \cdots, Z$ is called the *least equivalent set* of $X', Y', \cdots, Z'$. To indicate this equivalence we shall write

$$14.1 \qquad [X', Y', \cdots, Z'](\sim)[X, Y, \cdots, Z],$$

and, when convenient,

$$14.2 \qquad X'(\sim)X, \ Y'(\sim)Y, \cdots, Z'(\sim)Z.$$

14.3 *Theorem.* $(\sim)$ has the properties 4.12–4.17 of $\sim$.

The process of forming the least equivalent set is in general one of inflation. The complementary process of deflation is of equal importance for our purpose.

Consider first an M_n-number $(x_1, \cdots, x_n)$ different from ζ_n. Then there exists an integer h, $1 \leqq h \leqq n$, such that $x_h \neq 0$, $x_k = 0$, $k > h$, and h is called the *rank* of $(x_1, \cdots, x_n)$. If $x_n \neq 0$ the rank is n. By definition, the rank of ζ_n, for all integers $n > 0$, is 0.

14.4 *Definitions.* An M_n-number of rank $n>0$ is called an *M-number* of *order n.* The unique M-number of order 0 is (0), the *zero M-number.*

An M-number of order $n>0$ is thus an M_n-number of the form $(x_1, \cdots, x_n)$ with $x_n \neq 0$. Since an M-number of order n is also an M_n-number of order n, the definitions 11.2 of regularity and irregularity apply to M-numbers.

14.5 *Definitions.* If $(x_1, \cdots, x_n)$ is an M_n-number of rank h, the M-number $(x_1, \cdots, x_h)$ is called the *reduced equivalent M-number of* $(x_1, \cdots, x_n)$ or simply the *reduced equivalent* of $(x_1, \cdots, x_n)$. The reduced equivalent of ζ_n is (0).

Let $X', Y', \cdots, Z'$ be any set of k (k finite) M-numbers, and let (see 14.1)

$$[X', Y', \cdots, Z'](\sim)[X, Y, \cdots, Z].$$

Then $X, Y, \cdots, Z$ are all of the same order and the theory of Sections 11–13 can be applied to them.

15. *M-Varieties.* A variety whose elements are M-numbers is called an *M-variety.* The rational operations in an M-variety are defined as follows, with reference to Section 12.

Let $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ be M-numbers of any orders a, b, (a, b, by the definitions, are finite). Let

$$(x_1, \cdots, x_a)(\sim)(x_1', \cdots, x_c'),\ (y_1, \cdots, y_b)(\sim)(y_1', \cdots, y_c'),$$

as in 14.2, so that c is a or b if $a=b$ and the greater of a, b if $a \neq b$. The relation $(=)$ for M-numbers is then defined thus:

$$(x_1, \cdots, x_a)(=)(y_1, \cdots, y_b)$$

if and only if $(x_1', \cdots, x_c')(=)_c(y_1', \cdots, y_c')$, and hence if and only if $a=b$.

By 12.1,

$$(x_1', \cdots, x_c')(+)_c(y_1', \cdots, y_c')(=)_c(x_1' + y_1', \cdots, x_c' + y_c').$$

Let the reduced equivalent (14.5) of $(x_1' + y_1', \cdots, x_c' + y_c')$ be $(s_1, \cdots, s_e)$. Then the *M-sum* $(x_1, \cdots, x_a)(+)(y_1, \cdots, y_b)$ of the M-numbers $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ is defined to be the M-number $(s_1, \cdots, s_e)$, and we write

$$(x_1, \cdots, x_a)(+)(y_1, \cdots, y_b)(=)(s_1, \cdots, s_e). \tag{15.1}$$

Similarly, the M-difference is defined from 12.2, and the M-product from 12.3:

$$(x_1, \cdots, x_a)(-)(y_1, \cdots, y_b)(=)(d_1, \cdots, d_f); \tag{15.2}$$

$$(x_1, \cdots, x_a)(\cdot)(y_1, \cdots, y_b)(=)(p_1, \cdots, p_g), \tag{15.3}$$

where $(d_1, \cdots, d_f)$ is the reduced equivalent of $(x_1' - y_1', \cdots, x_c' - y_c')$, and $(p_1, \cdots, p_g)$ is the reduced equivalent of $(x_1' y_1', \cdots, x_c' y_c')$.

It is clear that if we proceed similarly to define M-quotients, such exist when and only when the two M-numbers concerned are of the same order and moreover the divisor $(z_1, \cdots, z_n)$ as in 12.4 is (as there) regular.

(15.4) Division for M-numbers is defined by (12.4) in which $(x_1, \cdots, x_n)$, $(z_1, \cdots, z_n)$ are M-numbers of order n.

The last brings up the important question of "division" for irregular divisors, to which we return in Section 17.

15.5 *Theorem.* The theorems 12.5–12.8 imply the corresponding theorems with M_n, $(\ )_n$, and ζ_n replaced by M, $(\ )$, and (0) respectively.

We have thus constructed ova, commutative semigroups, rays and rings whose elements are M-numbers. These are called *M-varieties.* In other words, we have constructed the commoner algebraic varieties for one-rowed matrices when the numbers of coordinates of the matrices combined according to the rational operations are not necessarily equal. For our purpose it is unnecessary to generalize this to rectangular matrices. The generalization can be attained in several ways, some of which will be considered on another occasion.

The details of the proofs of the several parts of 15.5 are all simple and may be omitted; they are in fact immediate consequences of the definitions. To combine several M-numbers by rational operations, we first replace them by their least equivalent set. Having done so, we operate with the equivalents as in Section 12 and replace the unique M_n-number which results by its reduced equivalent M-number.

The varieties in 15.5 are called *M-varieties.* Thus we shall speak of an M-ring, etc.

16. *Ordered M-varieties.* We may refer here to Sections 13, 15.

16.1 *Definition.* If $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ are any M-numbers, and

$$(x_1, \cdots, x_a)(\sim)(x_1', \cdots, x_c'),\ (y_1, \cdots, y_b)(\sim)(y_1', \cdots, y_c'),$$

as in Section 15, we say that $(x_1, \cdots, x_a)(>)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(>)_c(y_1', \cdots, y_c')$; that $(x_1, \cdots, x_a)(<)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(<)_c(y_1', \cdots, y_c')$, and that $(x_1, \cdots, x_a)(=)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(=)_c(y_1', \cdots, y_c')$.

The last, $(=)$, implies that $c=a=b$.

16.2 *Theorem.* The ring of all M-numbers in 15.5 is ordered with respect to $(>)$.

The necessary proofs are obvious from the last of Section 15 and 13.3.

17. *Proper Divisibility of M-numbers.* Here we may refer to 15.4.

17.1 *Definitions.* If $(x_1, \cdots, x_n)$ is an irregular M-number other than (0), and if $x_j=0(j=j_1, j_2, \cdots, j_s)$, where $j_1<j_2<\cdots<j_s$, and $x_h\neq 0(h\neq j_1, j_2, \cdots, j_s)$, the regular M-number $(j_1, j_2, \cdots, j_s)$ is called the *index of irregularity* of $(x_1, \cdots, x_n)$.

By definition, the index of irregularity of (0) is (0), and the index of irregularity of a regular M-number is 0.

Irregular M-numbers of the same order having the same (in the sense of $(=)$) index of irregularity, are said to be *co-irregular*, as also are regular M-numbers of the same order.

17.2 *Proper Divisors.* We remark once for all that division by (0) is not

defined. From the definitions 17.1 it is seen that if $(x_1, \cdots, x_n)$ is an irregular M-number different from (0), there exist an infinity of irregular M-numbers $(y_1, \cdots, y_m)$ other than (0) such that $(x_1, \cdots, x_n)(\cdot)(y_1, \cdots, y_n)(=)(0)$. For example, $(0, 2, 3, 0, 1)(\cdot)(y_1, 0, 0, y_2)(=)(0)$, where y_1, y_2 are arbitrary (finite) real numbers. Thus irregular M-numbers are "divisors of zero", the "zero" being the zero element (modulus of addition) of the system. Again, if $(z_1, \cdots, z_n)$ is any irregular M-number other than (0), there exist infinitely many pairs of M-numbers, $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, neither (0), at least one of which is necessarily irregular, such that $(x_1, \cdots, x_n(\cdot)(y_1, \cdots, y_n)(=)(z_1, \cdots, z_n)$.

17.3 *Theorem.* If $(x_1, \cdots, x_n)$, $(z_1, \cdots, z_n)$ are co-irregular M-numbers different from (0), there exists a unique M-number $(y_1, \cdots, y_n)$ co-irregular with them, such that

$$(x_1, \cdots, x_n)(\cdot)(y_1, \cdots, y_n)(=)(z_1, \cdots, z_n).$$

17.4 *Theorem.* The M-product $(\cdot)$ of two (and hence of any finite number) of co-irregular M-numbers is co-irregular with each of the factors.

17.5 *Definitions.* The irregular M-number $(y_1, \cdots, y_n)$ uniquely determined in 17.3 is called the *M-quotient* of $(z_1, \cdots, z_n)$ by $(x_1, \cdots, x_n)$, and we write

$$(y_1, \cdots, y_n)(=)(z_1, \cdots, z_n)(\div)(x_1, \cdots, x_n);$$

$(x_1, \cdots, x_n)$ is said to divide $(z_1, \cdots, z_n)$ *properly*.

18. *Infinite M-numbers.* Thus far we have considered only one-rowed matrices having a finite number of real, finite coordinates. To develop the theory of unique decomposition for a variety containing a countable infinity of distinct elements it is necessary to introduce matrices having a countable infinity of coordinates. Such matrices fall into two classes, finite and infinite, the second of which, being of no use for our purpose here, will be ignored.

18.1 *Definitions.* A one-rowed matrix $(x_1, x_2, \cdots, x_n, \cdots)$, whose coordinates are real, finite numbers is said to be *finite* if and only if there exists a finite integer m such that $x_h=0$ for all integers $h>m$. If $(x_1, \cdots, x_n, \cdots)$, is of infinite order and finite, it is called an M_∞-number. The *zero* M_∞-number Ω is the M_∞-number all of whose coordinates are zero.

Operations upon M_∞-numbers are now referred to abstractly identical operations upon M-numbers by making Ω, (0) correspondents, and taking as the correspondent of the M_∞-number $(x_1, \cdots, x_n, \cdots)$, other than Ω the M-number $(x_1, \cdots, x_m)$, where m is the least integer such that $x_m \neq 0$, $x_h=0$ for all integers $h>m$. Precisely as in defining operations and relations $(\ )$ for M-numbers from the corresponding operations and relations $(\ )_n$ for M_n-numbers, we do the like for $(\ )_\infty$ and M_∞-numbers with reference to $(\ )$ and M-numbers.

18.2 *Definitions.* The *reduced equivalent* of Ω is the M-number (0); the *reduced equivalent* of the M_∞-number $(x_1, \cdots, x_n, \cdots)$, other than Ω is the M-number $(x_1, \cdots, x_m)$, where m is the least integer such that $x_m \neq 0$, $x_h=0$ for all integers $h>m$. We write $\Omega(\sim)_\infty(0)$,

$$(x_1, \cdots, x_n, \cdots)(\sim)_\infty(x_1, \cdots, x_m).$$

We define the relations $(\gtrsim)_\infty$ by

$$(x_1, \cdots, x_n, \cdots)(\gtrsim)_\infty(y_1, \cdots, y_n, \cdots)$$

when and only when

$$(x_1, \cdots, x_n, \cdots)(\sim)_\infty(x_1, \cdots, x_m),\ (y_1, \cdots, y_n, \cdots)(\sim)_\infty(y_1, \cdots, y_s),$$
$$(x_1, \cdots, x_m)(\gtrsim)(y_1, \cdots, y_s),$$

respectively.

Refer now to 15.1–15.4 and let the respective reduced equivalents of the M_∞-numbers,

$$(x_1, \cdots, x_a, \cdots), \cdots, (p_1, \cdots, p_g, \cdots),\ (x_1, \cdots, x_n, \cdots), (z_1, \cdots, z_n, \cdots)$$

be the M-numbers,

$$(x_1, \cdots, x_a), \cdots, (p_1, \cdots, p_g), \cdots, (x_1, \cdots, x_n),\ (z_1, \cdots, z_n).$$

Then the rational operations upon M_∞-numbers are defined thus:

$$(x_1, \cdots, x_a, \cdots)(+)_\infty(y_1, \cdots, y_b, \cdots)(=)_\infty(s_1, \cdots, s_e, \cdots)$$

when and only when 15.1 holds, and similarly for $(-)_\infty$ and 15.2 $(\cdot)_\infty$ and 15.3, $(\div)_\infty$ and 15.4.

18.3 *Theorem.* The theorems in 15.5 referring to M-numbers and operations ( ) imply the corresponding theorems for M_∞-numbers and operations $(\ )_\infty$, with M, ( ), (0) replaced by M_∞, $(\ )_\infty$ and Ω respectively, and any M-number $(x_1, \cdots, x_n)$ other than (0) replaced by the M_∞-number $(x_1, \cdots, x_n, \cdots)$.

In the same way from 16.2 we have

18.4 *Theorem.* The ring of all M_∞-numbers is ordered with respect to $(>)_\infty$.

18.5 *Definition.* Two or more M_∞-numbers are said to be *co-irregular* when their reduced equivalents (M-numbers) are co-irregular (Section 17); the *index of irregularity* of an M_∞-number is defined to be the index of irregularity of its reduced equivalent.

18.6 *Theorem.* If $(x_1, \cdots, x_n, \cdots)$, $(z_1, \cdots, z_n, \cdots)$, are co-irregular M_∞-numbers different from Ω, there exists a unique M_∞-number $(y_1, \cdots, y_n, \cdots)$, co-irregular with them, such that

$$(x_1, \cdots, x_n, \cdots)(\cdot)(y_1, \cdots y_n, \cdots)(=)_\infty(z_1, \cdots, z_n, \cdots).$$

The M_∞-product of any finite number of co-irregular M_∞-numbers is co-irregular with each of the factors.

This is evident from 18.5, 17.3, 17.4.

18.7 *Definitions.* The irregular M_∞-number $(y_1, \cdots, y_n, \cdots)$, uniquely determined in 18.6 is called the M_∞-quotient of $(z_1, \cdots, z_n, \cdots)$, by $(x_1, \cdots, x_n, \cdots)$, and we write

$$(y_1, \cdots, y_n, \cdots)(=)_\infty(z_1, \cdots, z_n, \cdots)(\div)_\infty(x_1, \cdots, x_n, \cdots);$$

$(x_1, \cdots, x_n, \cdots)$, is said to divide $(z_1, \cdots, z_n, \cdots)$, *properly.*

18.8 *Scalar Products, Derivatives, Etc.* Although what follows is not required for the arithmetical problem in view, it is included to complete the algebraic part of the theory. As the developments thus far for M_n-, M-, and M_∞- numbers are abstractly identical, we may discuss all simultaneously, and lay down the following comprehensive definition of the subsequent notation.

18.81 *Definition.* N shall refer to any one of M_n, M, M_∞, the corresponding operations and relations $(\ )_n$, $(\ )$, $(\ )_\infty$ in the three cases being subsumed under the common one in the third line of the table in Section 10, with any one of the three interpretations $(\ )_n$, $(\ )$, $(\ )_\infty$ understood throughout a given context. The elements of any one of the three species of varieties will be called *N-numbers*, and will be denoted by capital Latin letters, $A, B, \cdots, X, Y, \cdots$. The modulus of the operation now denoted by $+$ will be denoted by 0 (with, of course, the interpretation appropriate to the particular species concerned).

18.82 *Definition.* If c is a real or complex number, the scalar product cX of c and the N-number X is the matrix obtained from X by multiplying each of the coordinates of X by c. If c is pure imaginary, cX is called an *imaginary N-number*. (If c is real, cX is an N-number).

18.83 *Theorem.* Scalar products as in 18.82 obey all the laws of combination of scalar products in a linear algebra.

For example, $(a+b)X=aX+bX$, where $a+b$ is ordinary addition, and, by 18.81, the $+$ on the right is interpreted in the N-variety.

Let $[X', Y'](\sim)[X, Y]$ as in 14.1, so that X, Y are of the same order n. Then if the jth coordinate y_j of Y is a function of the jth coordinate x_j of $X(j=1, \cdots, n)$, Y' is said to be a function of X'.

If Y' is a function of X', and if the derivatives dy_j/dx_j exist $(j=1, \cdots, n)$, the *derivative of Y' with respect to X'* is the matrix whose jth coordinate is dy_j/dx_j. Anti-derivatives, integrals as sums, derivatives of complex N-functions and their integrals are defined in a similar manner; namely, by means of equivalent pairs of N-numbers, the definition from the real or complex field being applied to the individual coordinates of the pair having the same order. We shall not stop to state the procedure for irregular N-numbers where it differs from that for regular, as it is sufficiently evident from the like for proper division as already constructed.

The analyses then defined are abstractly identical with those of the real and complex number systems.

V. ARITHMETIC OF N-NUMBERS

19. *N-integers.* For the significance of N see 18.81; for the definition of integers and primes, see 3.1–3.3, and for proper divisibility, 17.5. From these the complete rational arithmetic, isomorphic to that of the rational integers, of N-numbers is readily constructed. It will be sufficient to lay down the cardinal definitions and to state a few of the principal elementary theorems.

19.1 *Definitions.* An N-number all of whose coordinates are rational integers is called an N-integer.

If A, B are N-integers, A is said to *divide* B, written $A \mid B$, if and only if B is properly divisible by A (17.5) and the N-quotient (17.5) of B by A is an N-integer. If $A \mid B$, A is called an *N-divisor* of B.

This is the definition of *arithmetical* divisibility as distinguished from mere *algebraic* divisibility as in 17.5.

An N integer, other than the zero N-integer, each of whose non-zero coordinates $=1$ in absolute value, is called an *N-unit.*

Each of the coordinates of an N-unit is therefore one of the numbers $-1, 0, 1$, the case where each is 0 excluded.

We now separate N-integers into four mutually exclusive classes:

19.11 The zero *N-integer* (0).

19.12 *Positive N-integers*: an N-integer is said to be *positive* if each of its coordinates is >0.

19.13 *Mixed N-integers*: an N-integer is said to be *mixed* if at least one, and not all, of its coordinates are negative.

19.14 *Negative N-integers*: an N-integer is said to be *negative* if each of its coordinates is <0.

19.2 *Theorem.* If X is an N-integer other than (0), there exists precisely one N-unit U such that the N-quotient of X by U is positive.

Thus it suffices to discuss divisibility of N-integers for positive N-integers only.

The *N-greatest common divisor* (abbreviated N. G. C. D) of the positive N-integers, A, B is that positive divisor which is divisible by every common positive divisor of A, B; the *N-least common multiple* (N. L. C. M) of A, B is that positive common multiple of A, B which divides every common multiple of A, B.

It is easily verified that the terms greatest and least have here the significance abstractly identical with the like for G. C. D and L. C. M in rational arithmetic: if G is the N. G. C. D of A, B and D is any common positive divisor, then (see 18.82), $G \geqq D$; if L is the N. L. C. M, and M is any common positive multiple of A, B, then $L \leqq M$. Further, all the properties of G. C. D and L. C. M have their N-isomorphs which can be written down at once from the comprehensive correspondence established between N-integers and rational integers; for example $AB = GL$.

If P is an N-integer other than the N-zero or an N-unit, and if P is positive and has only the divisors P and an N-unit, P is called a (positive) N-prime.

19.3 *Theorem.* If an N-prime divides the product of two positive N-integers it divides at least one of the factors.

19.4 *Theorem.* An N-integer is divisible by only a finite number of N-primes.

19.5 *Theorem.* A positive N-integer is, apart from permutations of the factors, uniquely a product of positive N-primes.

If the N. G. C. D of the positive N-integers A, B is an N-unity, A, B are said to be *coprime.* Let $x_j (j = i_1, i_2, \cdots, i_s)$ be all those coordinates of the positive N-integer X which are >0 so that $\nu(X) \equiv x_{i_1} \cdots x_{i_s}$ is the *norm* of X. Each of

the $\nu(X)$ N-integers X' obtained from X by substituting for x_j in turn the rational integers x_j', where $0<x_j' \leqq x_j (j=i_1, \cdots, i_s)$, is co-irregular with X. Consider the special case where each $x_j>1$, and let η be the positive N-unit co-irregular with X. Then, if ϕ denotes Euler's function, there are precisely $\phi(x_{i_1}) \cdots \phi(x_{i_s})$, say $\Phi(X)$ of the $\nu(X)$ integers X' co-irregular and coprime with X. Let A be any positive N-integer coprime with X. Then, if A^n denotes the N-product of precisely n factors each A, the N-difference of $A^{\Phi(X)}$ and η is N-divisible by X, which is the analogue for N-numbers of Euler's extension of Fermat's theorem. The complete theory of congruences requires several details on account of irregularity into which we need not enter here.

It is to be noticed in the ordering of N-numbers that $\gamma \rightarrow \mu_*$ in 8.45 is insufficient to ensure $\alpha \dagger \gamma \rightarrow \beta \dagger \gamma$ *unless* it be interpreted in the *narrow* sense that γ is *positive*. Since a rational integer which is positive in the ordinary sense is an N-integer which is positive as defined in 19.12, the isomorphism is complete. Similarly the isomorphism is exact for all N-numbers when in 19.12 for N-integer we substitute N-number (as a definition).

VI. ARITHMETIZATION OF A COMMUTATIVE VARIETY

20. *Finite Decomposition.* Let Γ be a commutative variety (Section 4) having a countable (Section 2, end) subset Σ of elements (Γ, Σ may or may not be identical), such that there exists in Σ a subset Π of elements having with respect to the composition $*$ of Γ the following properties.

20.1 No element of Π is a composite of elements of Π.

20.2 Every element of Σ not in Π is a composite of a finite number of elements of Π in one way only apart from permutations of elements.

In 20.2 we presuppose the convention

20.3 If Γ has the modulus μ_*, and α is any element of Γ, $\mu_* * \alpha$ (and hence also $\alpha * \mu_*$) is to be replaced by α wherever it occurs.

Since Γ is countable, it may be ordered. Assume any ordering of its elements —one usually will be given by some special property of Γ which is simpler than the others, of which an infinity are possible if Γ contains an infinity (countable) of distinct elements. Hence also the elements $\pi_j (j=1, 2, \cdots)$, of Π are ordered; say that $\pi_j \rightarrow \pi_k$ (see §8) when $j>k$.

20.4 We shall say that the element α of Σ has a *finite* decomposition if and only if

$$\alpha \sim \pi_{j_1} * \pi_{j_2} * \cdots * \pi_{j_s},$$

where each of $j_1, \cdots, j_s$ is finite and s is finite; and decomposition shall mean *finite decomposition.*

A composite of precisely n elements ($n<0$) each of which is β will be written β^n, and β^0 shall denote μ_*. With these conventions the unique decomposition of α may be written

$$\alpha \sim \pi_1^{x_1} * \pi_2^{x_2} * \cdots * \pi_n^{x_n},$$

where the $x_j (j=1, \cdots, n, \cdots)$ are rational integers $\geqq 0$, not all 0. We shall

call the N-number $(x_1, x_2, \cdots, x_n, \cdots)$, the *index* of α, and we shall write $I(\alpha) \equiv (x_1, \cdots, x_n, \cdots)$; $I(\mu_*) = 0$. (See 18.82.)

21. *Construction of Varieties from* Γ. Let small Greek letters $\alpha, \beta, \cdots$, denote elements of Γ, and let ∇ denote any one of the symbols of operations and relations in a given N-variety (18.87). Then the variety Γ_N, whose elements coincide with those of Γ, is completely defined by the following definition of ∇_N:

21.1 $\alpha \nabla_N \beta$ when and only when $I(\alpha) \nabla I(\beta)$.

22. *Arithmetic of* Γ. This is completely defined by

22.1 The integers of Γ are all those elements α of Γ for which $I(\alpha)$ is an N-integer. (This is implied by previous definitions.)

22.2 A property $P(\alpha, \beta, \cdots)$, of integers of Γ subsists when and only when in the N-variety in 21.1 the property $P(I(\alpha), I(\beta), \cdots)$, subsists between the N-integers $I(\alpha), I(\beta), \cdots$.

For example, $\alpha \,|\, \beta$, that is, α divides β arithmetically in Γ when and only when $I(\alpha) \,|\, I(\beta)$ in the N-variety; $\alpha > \beta$ in Γ when and only when $I(\alpha) > I(\beta)$; $\alpha + \beta$ in Γ is equal to γ, where γ is the uniquely determined element of Γ such that $I(\gamma) = I(\alpha) + I(\beta)$ in the N-variety.

Owing to the length of this paper we must omit applications. If the N-variety is a ring, addition in Γ is multiplication in the ring; multiplication in Γ is a new process in the ring, closely connected with the G. C. D and L. C. M in the ring and with the properties of elements of the ring considered as decomposed into powers of coprime products of distinct primes. It is interesting that the arithmetical properties (Fermat's theorem, quadratic reciprocity, etc.) of this new process are abstractly identical with those of multiplication and arithmetical divisibility.

SEMI-SERIAL ORDER

By ALBERT A. BENNETT, Brown University

The subjects of serial order and of the calculus of classes have proved particularly attractive to postulationalists. In each case by use of relatively few postulates the complete logical basis for an entire mathematical discipline may be exhibited. Excellent (but entirely unrelated) discussions of both of these topics have been provided by E. V. Huntington.[1] The two subjects differ considerably but both may be developed by use of a common symbol, $<$, of order relation. Some other important systems differing from both show also an essentially analogous use of a symbol of dyadic ordinal relation (e.g. the system of idempotent elements in some algebras, see example 12, below). It appears therefore worth noting that a body of common relations found in these various basic mathematical studies has hitherto escaped a common formulation.

[1] *The Continuum and Other Types of Serial Order*, 2nd Edition, 1917, Harvard University Press, *Postulates for the Algebra of Logic*, Transactions of the American Mathematical Society, vol. 5 (1904), pp. 288–309. (In particular, "The second set," pp. 297–305.)

A class K with at least two distinct elements will be said to have *semi-serial order* if it satisfies postulates I–V given below. In many interesting cases postulates VI–VIII are also satisfied. The wording used, save for VIII, will be essentially that of E. V. Huntington, for corresponding postulates in the algebra of logic. However by the omission of certain postulates there given but here extraneous and by introducing VIII an essentially new system of more extensive application is obtained. The arrangement of the postulates here used differs from Huntington's arrangement.

K is a class of elements. $\leqq$ is a dyadic relation among elements of K.

I. (Reflexiveness.) $a \leqq a$ whenever a belongs to the class.

II. (Restricted Symmetry.) If $a \leqq b$, and also $b \leqq a$, then $a = b$.

III. (Transitivity.) If $a \leqq b$, and $b \leqq c$, then $a \leqq c$.

IV. (Predecessor.) If $a \neq b$, and if neither $a \leqq b$, nor $b \leqq a$, then there is an element m, such that

(1) $m \leqq a$, and $m \leqq b$,

(2) if $x \neq m$, is such that $x \leqq a$, and $x \leqq b$, then $x \leqq m$.

V. (Successor.) If $a \neq b$, and if neither $a \leqq b$, nor $b \leqq a$, then there is an element n, such that

(1) $a \leqq n$, and $b \leqq n$, and

(2) if $y \neq n$, is such that $a \leqq y$, and $b \leqq y$, then $n \leqq y$.

VI. (Initial Element.) There is an element i such that for each $a \neq i$, $i \leqq a$.

VII. (Ultimate Element.) There is an element u such that for each $a \neq u$, $a \leqq u$.

VIII. (Primitive Element.) If there is an element i such that for each $a \neq i$, $i \leqq a$, then there is at least one element p, $\neq i$, such that if $t \leqq p$, then $t = i$ or $t = p$.

That these postulates I–VII are independent has been shown by Huntington for a different purpose. That VIII is at least ordinally independent of I–VII is seen by taking the example of the set of rational numbers between zero and unity, ends inclusive.

Deductions[2] *from postulates* I–V:

Th. 1. If $a = b$, $a \leqq b$. (I)

Def. 1. If $a \leqq b$, and $a \neq b$, then the notation $a < b$ may be used.

Th. 2. If $a \neq b$ then not both $a \leqq b$, and $b \leqq a$ are possible simultaneously. (II, Def. 1)

Th. 3. If simultaneously $a_1 \leqq a_2, a_2 \leqq a_3, \cdots, a_{k-1} \leqq a_k$, ($k$ finite) then $a_1 \leqq a_k$. (III)

Cor. If $a_1 \leqq a_2$, $a_2 \leqq a_3, \cdots, a_{k-1} \leqq a_k$ but not $a_1 < a_k$, then $a_1 = a_2 = \cdots = a_{k-1} = a_k$ (II, III)

Th. 4. The m of IV is unique. (II, IV)

Hence the rule of combination, $\wedge$, may be defined[3] as follows:

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[2] These save for necessary rearrangements and a few additions are essentially those of Huntington, l.c.

[3] This notation in the case of subalgebras is traditional. Cf. L. E. Dickson, *Algebras and their Arithmetics*. University of Chicago, 1923, p. 26. It suggests the $\frown$ used by Peano, Russell, and later writers for logical product of classes.

Def. 2. If $a \leqq b$, the element $a \wedge b$ is defined as a. If $b \leqq a$ the element $a \wedge b$ is defined as b. (If $b = a$, then $a \wedge a$ is defined as a.) Otherwise $a \wedge b$ is defined as the m of IV.

Cor. 1. $a \wedge b = b \wedge c$. (II, IV)

Cor. 2. $(a \wedge b) \leqq a$, $(a \wedge b) \leqq b$. (I, II, IV)

Cor. 3. If $x \leqq a$, and $x \leqq b$, then $x \leqq (a \wedge b)$. (I, II, IV)

Th. 5. The n of V is unique. (II, V)

Hence the rule of combination, $\vee$, may be defined as follows.[4]

Def. 3. If $a \leqq b$, the element $a \vee b$, is defined as b. If $b \leqq a$, the element $a \vee b$ is defined as b. (If $b = a$, then $a \vee a$ is defined as a.) Otherwise $a \vee b$ is defined as the n of V.

Cor. 1. $a \vee b = b \vee a$. (II, V)

Cor. 2. $a \leqq (a \vee b)$, $b \leqq (a \vee b)$. (I, II, V)

Cor. 3. If $a \leqq y$, and $b \leqq y$, then $(a \vee b) \leqq y$. (I, II, V)

Th. 6. If $x \leqq (a \wedge b)$, then $x \leqq a$, and $x \leqq b$. (I, II, III, IV)

Th. 7. If $(a \vee b) \leqq y$, then $a \leqq y$, and $b \leqq y$. (I, II, III, V)

Th. 8. If $a \leqq b$, and $x \leqq y$, then $(a \wedge x) \leqq (b \wedge y)$. (I, II, III, IV)

In particular if $x \leqq y$, then $(a \wedge x) \leqq (a \wedge y)$.

Th. 9. If $a \leqq b$, and $x \leqq y$, then $(a \vee x) \leqq (b \vee y)$. (I, II, III, V)

In particular if $x \leqq y$, then $(a \vee x) \leqq (b \vee y)$.

Th. 10. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. (I, II, III, IV)

Def. 4. The unique common element $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ may be denoted by $a \wedge b \wedge c$.

Th. 11. $(a \vee b) \vee c = a \vee (b \vee c)$. (I, II, III, V)

Def. 5. The unique common element $(a \vee b) \vee c = a \vee (b \vee c)$ may be denoted by $a \vee b \vee c$.

Th. 12. If $a_1, a_2, \cdots, a_k$ are any k elements of the class there is a unique element m which may be denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ and which is such that

(1) $m \leqq a_1$, $m \leqq a_2, \cdots, m \leqq a_k$ and

(2) if x is such that $x \leqq a_1$, $x \leqq a_2, \cdots, x \leqq a_k$, then $x \leqq m$.

(I, II, III, IV)

Th. 13. If $a_1, a_2, \cdots, a_k$ are any k elements of the class there is a unique element n which may be denoted by $a_1 \vee a_2 \vee \cdots \vee a_k$, and which is such that

(1) $a_1 \leqq n$, $a_2 \leqq n, \cdots, a_k \leqq n$, and

(2) if y is such that $a_1 \leqq y$, $a_2 \leqq y, \cdots, a_k \leqq y$, then $n \leqq y$. (I, II, III, V)

Proofs of Theorems 8, 10:

The other theorems given above follow immediately from the postulates. The proofs of theorems 8 and 10 are here given as by Huntington. Theorems 9, 11, may be proved by symmetry.

Proof of Th. 8: By Th. 6, $(a \wedge x) \leqq a$, and $(a \wedge x) \leqq x$. From $a \wedge x \leqq a$, and $a \leqq b$, follows by III that $(a \wedge x) \leqq b$. From $(a \wedge x) \leqq x$ and $x \leqq y$ follows by III that $(a \wedge x) \leqq y$. Hence by Th. 4, Cor. 3, $(a \wedge x) \leqq (b \wedge y)$, as desired.

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[4] Cf. $\cup$ used by Peano, Russell, and later writers for logical sum of classes.

Proof of Th. 10: Let $(a \wedge b) = d$, and $d \wedge c = e$. Then by Th. 4, Cor. 2, $e \leqq d$, $e \leqq c$. Also $d \leqq a, d \leqq b$. Hence $e \leqq a, e \leqq b, e \leqq c$. Hence by Th. 4, Cor. 3, $e \leqq (b \wedge c)$, and hence again $e \leqq a \wedge (b \wedge c)$. Thus $(a \wedge b) \wedge c \leqq a \wedge (b \wedge c)$. Similarly $a \wedge (b \wedge c) \leqq (a \wedge b) \wedge c$. Hence the theorem.

As stated previously any system with at least two elements, satisfying I–V, is said to have *semi-serial order*. In particular it may be noted that these postulates are satisfied by each system having serial order. Indeed for postulates of serial order we may take the following due to Huntington (l.c.).

0. The class K is not an empty class, nor a class containing merely a single element.

1. If a and b are distinct elements of K, then either $a < b$, or $b < a$.

2. If $a < b$, then a and b are distinct.

3. If $a < b$, and $b < c$, then $a < c$.

It may be noted that while I–VIII include the assumption that K contains at least two distinct elements, I–V fails to assure this property of K, here covered by 0. Postulates 2, and 3, are essentially equivalent to II and III. Postulate 1 is too strong for general semi-serial order. The postulates IV and V hold for serial order but in view of 1 they are only vacuously true. However the set of theorems 1–13, given above which for the general case make explicit use of IV and V, not only continue to hold in the special case of serial order but are not in most cases vacuous or trivial.

Further Deductions from I–VII:

Th. 14. The element i of VI is unique. (II, VI)

Th. 15. For each a, $i \leqq a$, and if $x \leqq i$, then $x = i$. (I, II, VI)

Th. 16. The element u of VII is unique. (II, VII)

Th. 17. For each a, $a \leqq u$, and if $u \leqq y$, then $y = u$. (I, II, VII)

Th. 18. $a \vee i = i$, and $a \wedge u = a$. (I, II, IV, VI, VII)

Th. 19. $a \vee i = a$, and $a \vee u = u$. (I, II, V, VI, VII)

The proofs of these are again immediate. Hitherto no use has been made of VIII, which is also the only postulate whose dual statement, (interchanging left and right members, and interchanging i and u), is not asserted. One conclusion, using only a small part of the content of VIII, is

Th. 20. $i \neq u$. (II, VI, VII, VIII)

The chief significance in VIII lies in the fact that it also is satisfied by the more interesting of the examples of semi-serial order as given below. Other analogous examples can be obtained with ease.

Some Examples Satisfying I-VIII:

1. The system of natural numbers 1, 2, 3, $\cdots$ together with ω, the first transfinite ordinal. Here $i = 1$, $p = 2$, $u = \omega$. By $a \leqq b$ is meant the usual order relation of magnitude.

2. The system of real non-negative numbers together with -1, and $+\infty$. Here $i = -1, p = 0, u = +\infty$. By $a \leqq b$ is meant the usual order relation of magnitude.

3. The system of non-negative rational integers. Here $i=1$, p is any prime number ($\neq 1$), $u=0$. By $a \leqq b$, is meant a is a divisor of b. In particular unity is counted as a divisor of each number of the system, and zero is regarded as divisible by each such number. Here $a \wedge b$ is the greatest common divisor, and $a \vee b$, the least common multiple of a and b.

4. The subclasses of a given class u. By i is meant the null class and by p, any class of but a single element. By $a \leqq b$, is meant that a is a subclass of b. By $a \wedge b$ is meant the logical product and by $a \vee b$ is meant the logical sum of a and b.

5. The linear projective subspaces of a given space u of N dimensions. By i is meant the null space. By p is meant a point. By $a \leqq b$ is meant that a is a linear subspace of b. By $a \wedge b$ is meant the intersection of a and b, by $a \vee b$ is meant the linear space of least dimensionality containing both a and b. Note that while $a \wedge b$ is the logical product $a \vee b$ is not the logical sum of the spaces regarded as classes of points. It may be remarked that the theory of the intersections of projective subspaces[5] may be regarded as more general than that of the theory of the projective space obtained by adjoining a suitable infinite region to a Euclidean space of N dimensions. The projective space may be the domain of a "finite geometry" in the sense of Veblen and Young.[6] While Veblen and Young explicitly stipulate that a projective line shall contain at least three points, the assumption that it shall contain only two points is not completely trivial. In fact the finite N-dimensional geometry where each line contains exactly two points is exactly the finite "logical field" discussed on its own account by Huntington. This relationship is however left unnoted by Veblen and Young, and by Huntington. Thus a complete theory for a finite geometry of N dimensions with $p+1$ points on a line, if it be so stated as to be applicable for $p=1$, gives in this special case the algebra of logic applied to a class of $N+1$ objects.

The transversal axiom assures that any generalized real projective space must contain in each real line a dense set of points. This notion of density is not essential to semi-serial order.

6. The set of closed intervals on a line, counting among intervals the null interval, i, each isolated point, p, and the entire line, u. By $a \leqq b$, is meant that a is a subinterval of b. By $a \wedge b$ is meant the common subinterval of a and b. By $a \vee b$ is meant the smallest interval containing both a and b. This is not in general the logical sum of the given intervals. One may note that for a corresponding system of segments (without end points) VIII will not be satisfied.

7. The set of all convex regions in a plane, counting among convex regions the null region, i, each isolated point p, and the entire plane u.

8. The set of all submodules of a given arithmetic module.[7]

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[5] Cf. P. H. Schoute, *Mehrdimensionale Geometrie.* 1. Teil, *Die Linearen Räume.* Leipzig, 1902. Eugenio Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi.* Piza, 1907.

[6] *Projective Geometry*, Vol. 1, 1910.

[7] Cf. F. S. Macaulay, *Modular systems*, Cambridge Tracts, No. 19, 1916.

9. The class of all linear sets[8] which are subsets of a given linear set.

10. The class of all regions in the plane each of which is bounded by a circle; including as circular region in particular the null region i, each region comprising but a single point p, and the entire plane u. Here $a \wedge b$ is the largest circular region contained entirely within the two given circular regions, and $a \vee b$ is the smallest circular region entirely covering the two given circular regions. Here $a \wedge b$ is less than the logical product of the two point-sets.

11. The system of subgroups of a given group u. By i is meant the identical element. By p is meant any cyclic subgroup of prime order. By $a \leqq b$ is meant a is a subgroup of b. By $a \wedge b$ is meant the maximal common subgroup of a and b, and by $a \wedge b$ is meant the minimal common supergroup of a and b. Since the entire discussion takes place within the group u, this minimal common supergroup always exists.

12. The set of idempotent elements (together with zero) in certain algebras of finite basis, containing a modulus as these terms are used by Dickson, (l.c.), where by $a \leqq b$ is meant that simultaneously $a \cdot a = a$, and $a \cdot b = b \cdot a = a$. By i is here meant the element 0, such that $0 \cdot 0 = 0$ and for each a, $a \cdot 0 = 0 \cdot a = 0$. By u is meant the element 1, (modulus) such that $1 \cdot 1 = 1$ and for each a, $a \cdot 1 = 1 \cdot a = a$. The elements here called primitive are those called primitive by Dickson (l.c. page 55).

COMBINING CONSTANT PROBABILITY FUNCTIONS[1]

By WILLIAM DOWELL BATEN, University of Michigan

The object of this paper is to show some interesting results which arise in combining probability functions which are constant in a particular interval and zero elsewhere, and to show how the probability function for the sum of n independent variables is formed and constructed. This will be treated for n finite and n infinite. The probability function for the sum, by a proper change of the variable, approaches the Gaussian function. The rapidity of approach is surprising for there is slight difference between the function for the sum of four variables and the Gaussian function.

The particular function treated comes up in considering the errors made in using any logarithmic table or any numerical table where the numbers are rounded-off numbers; also in considering measurements of time when the watch records time in units of seconds, tenths of seconds, etc. In geometric probability a constant probability function may arise. This particular function is a constant, that is regardless of the values of the variable the probability is the same in $(0, k)$ and the same for $(-\infty, 0)$ and (k, ∞). In a measure the probability is independent of the variable. Two applications are given at the close of the paper.

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[8] Used in the sense of *Dickson*, l.c., Chap. II.

[1] Read before the Michigan Section of the Mathematical Association of America, March 22, 1930.

As by-products, expressions for $n!$ and n^h, where $h=1, 2, \cdots, n-1$, are found. These expressions are certain summations which involve only integers.

The derivatives of the nth order will be used to indicate just how the different portions of the function for the sum differ at the ends of the intervals considered. Polynomials arise from which the general law for the sum is obtained.

Case 1

Let $f_1(x_1)dx_1$ be the probability that x_1 lies in the interval (x_1, x_1+dx_1), and $f_2(x_2)dx_2$ be the probability that x_2 lies in the interval (x_2, x_2+dx_2), where

$$f_1(x_1) = 0 \text{ in } (-\infty, 0); \qquad f_2(x_2) = 0 \text{ in } (-\infty, 0);$$

$$f_1(x_1) = \frac{1}{k} \text{ in } (0, k); \qquad f_2(x_2) = \frac{1}{k} \text{ in } (0, k);$$

$$f_1(x_1) = 0 \text{ in } (k, \infty); \qquad f_2(x_2) = 0 \text{ in } (k, \infty).$$

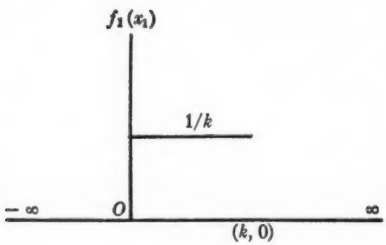

FIG. a

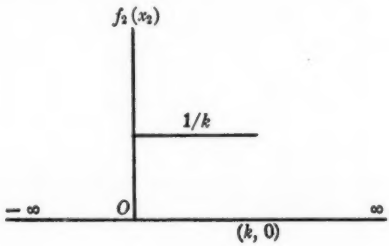

FIG. b

Let it be required to find the probability that $x_1+x_2=z$ lies in the interval $(z, z+dz)$. This can be secured by employing the following theorem which is well known:[1] *If $f(x)dx$ is the probability that x lies in the interval $(x, x+dx)$ and $g(y)dy$ is the probability that y lies in the interval $(y, y+dy)$, then the probability that $x+y=z$ lies in the interval $(z, z+dz)$ is*

$$F(z)dz = dz \int_{-\infty}^{\infty} f(x)g(z-x)dx.$$

There are two intervals which give results other than zero; they are the interval $0 \leqq z \leqq k$ and the interval $k \leqq z \leqq 2k$. According to the above theorem for $0 \leqq z \leqq k$,

$$F_2(z) = \int_{-\infty}^{0} 0 \cdot 0 dx_1 + \int_0^z \frac{1}{k} \cdot \frac{1}{k} dx_1 + \int_z^{\infty} \frac{1}{k} \cdot 0 dx_1 = \frac{z}{k^2};$$

[1] *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, by Von Karl Mayr in Wien. Monatshefte für Mathematik und Physik, vol. 30, art. 4, and *Calcul des Probabilités*, by H. Poincaré.

and for $k \leqq z \leqq 2k$,

$$F_2(z) = \int_{-\infty}^{0} 0 \cdot 0 dx_1 + \int_{0}^{z-k} \frac{1}{k} \cdot 0 dx_1 + \int_{z-k}^{k} \frac{1}{k} \cdot \frac{1}{k} dx_1 + \int_{k}^{\infty} 0 \cdot \frac{1}{k} dx_1 = \frac{2k-z}{k^2},$$

from which

$$F_2(z) = 0 \text{ in } (-\infty, 0),$$

$$= \frac{z}{k^2} \text{ in } (0, k),$$

$$= \frac{2k - z}{k^2} \text{ in } (k, 2k),$$

$$= 0 \text{ in } (2k, \infty).$$

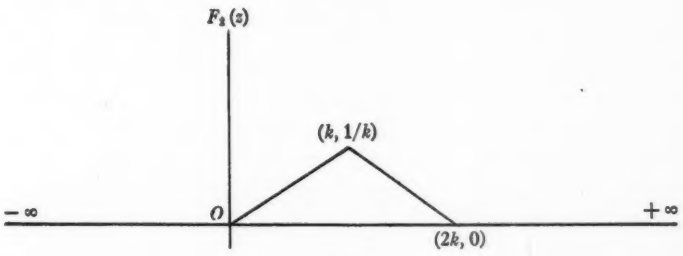

FIG. c

$F_2(z)$ is continuous for all values of z but its first derivative is discontinuous at $(0, 0)$, $(k, 1/k)$, and $(2k, 0)$. f_1 and f_2 are discontinuous at the origin and at the point with abscissa k, yet the probability function for the sum is continuous.

Let the probability function for the variable x_3 be $f_3(x_3)$ and similar to f_1 and f_2, that is equal to $1/k$ in $(0, k)$, and zero elsewhere. Then the probability function for the sum $x_1+x_2+x_3=z$ is given in five intervals: For $0 \leqq z \leqq k$,

$$F_3(z) = \int_{-\infty}^{0} 0 \cdot 0 dy + \int_{0}^{z} \frac{1}{k} \cdot \frac{z - y}{k^2} dy + \int_{z}^{\infty} \frac{1}{k} \cdot 0 dy = \frac{z^2}{2k^3};$$

for $k \leqq z \leqq 2k$,

$$F_3(z) = \int_{-\infty}^{0} 0 \cdot 0 dy + \int_{0}^{z-k} \frac{1}{k} \cdot \frac{2k - z + y}{k^2} dy + \int_{z-k}^{k} \frac{1}{k} \cdot \frac{z - y}{k^2} dy + \int_{k}^{\infty} 0 \cdot \frac{1}{k} dy$$

$$= \frac{6kz - 3k^2 - 2z^2}{2k^3};$$

for $2k \leqq z \leqq 3k$,

$$F_3(z) = \int_{-\infty}^{0} 0 \cdot 0 dy + \int_{0}^{z-2k} 0 \cdot (\ ) dy + \int_{z-2k}^{k} \frac{1}{k} \cdot \frac{2k - z + y}{k^2} dy + \int_{k}^{\infty} 0 \cdot (\ ) dy = \frac{9k^2 - 6kz + z^2}{2k^3};$$

where $y = x_1 + x_2$; from which

$$\begin{aligned} F_3(z) &= 0, \text{ in } (-\infty, 0), \\ &= z^2/2k^3, \text{ in } (0, k), \\ &= (-2z^2 + 6kz - 3k^2)/2k^3, \text{ in } (k, 2k), \\ &= (z^2 - 6kz + 9k^2)/2k^3, \text{ in } (2k, 3k), \\ &= 0, \text{ in } (3k, \infty). \end{aligned}$$

$F_3(z)$ is continuous and its first derivative is continuous for every value of z, yet its second derivative is discontinuous at the points whose abscissas are 0, k, $2k$, and $3k$.

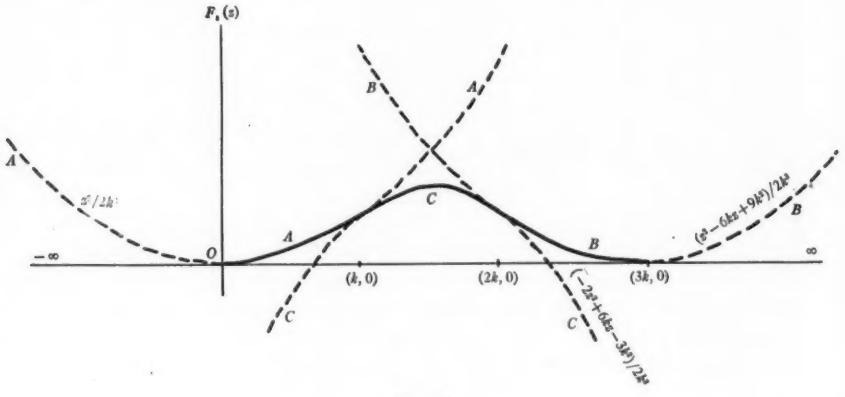

FIG. 1

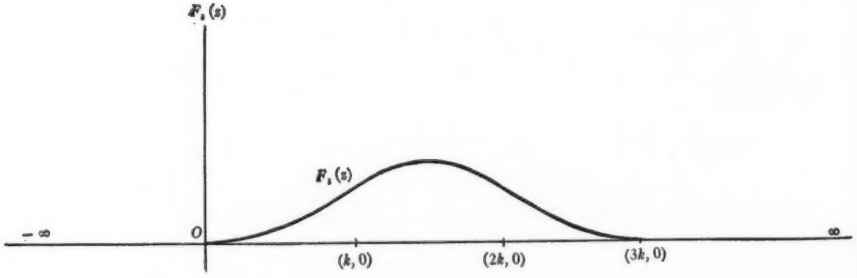

FIG. 2

Figure 1 shows the curves, parts of which form $F_3(z)$; while figure 2 shows $F_3(z)$. The dotted lines in figure 1 are not used to make up $F_3(z)$. At the points $(0, 0)$, $(k, 1/2k)$, $(2k, 1/2k)$, and $(3k, 0)$ the separate parts or curves are tangent.

If $f_4(x_4)$ is similar to f_1, then the probability function for the sum $x_1+x_2+x_3+x_4=z$, according to the theorem, is

$$F_4(z) = \frac{1}{3!k^4}(z^3) \text{ in } (0, k),$$

$$= \frac{1}{3!k^4}(-3z^3 + 12kz^2 - 12k^2z + 4k^3) \text{ in } (k, 2k),$$

$$= \frac{1}{3!k^4}(3z^3 - 24kz^2 + 60k^2z - 44k^3) \text{ in } (2k, 3k),$$

$$= \frac{1}{3!k^4}(-z^3 + 12kz^2 - 48k^2z + 64k^3) \text{ in } (3k, 4k),$$

$$= 0 \text{ for } (-\infty, 0) \text{ and } (4k, \infty).$$

$F_4(z)$ here is made up of six parts each represented by a different equation. $F_4(z)$ is continuous, its first and second derivatives are continuous for all values of z, but its third derivative is discontinuous at the points whose abscissas are 0, k, $2k$, $3k$, and $4k$. The different parts deviate less from the adjacent parts than was found in the probability function for the sum of three variables, because here the function and its first two derivatives are continuous for all values of z. The following figures shows how $F_4(z)$ is formed and $F_4(z)$ respectively. The

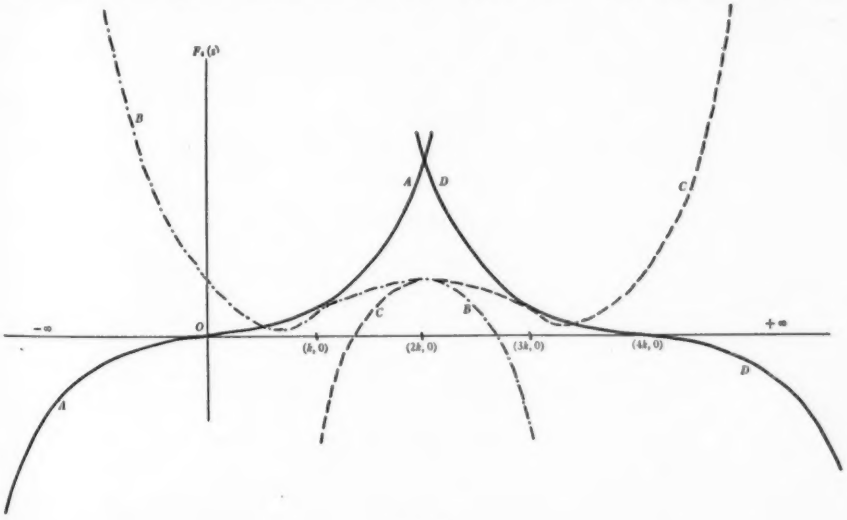

FIG. 3

parts coincide almost so completely that it is difficult to draw and indicate in the illustration how they fail to coincide perfectly. Figure 3 shows in a measure just this fact, while figure 4 appears to be one curve with all of its derivatives continuous.

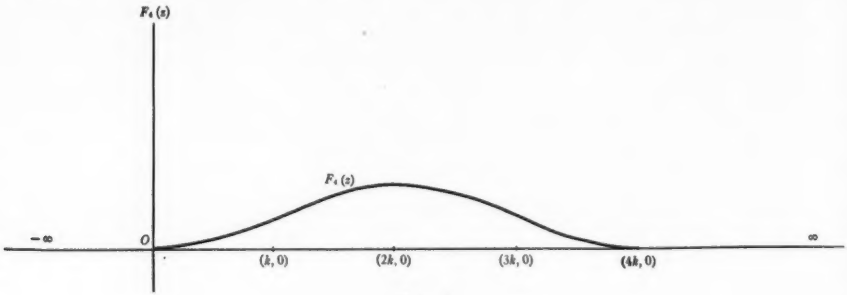

FIG. 4

Summarizing the results it is seen that for the proper intervals:

$$F_1(z) = \left\{\frac{1}{k}\right\}$$

$$F_2(z) = \left\{\begin{matrix}\frac{1}{k^2}(z) \\ \frac{1}{k^2}(-z + 2k)\end{matrix}\right\}$$

$$F_3(z) = \left\{\begin{matrix}\frac{1}{2!k^3}(z^2) \\ \frac{1}{2!k^3}(-2z^2 + 6kz - 3k^2) \\ \frac{1}{2!k^3}(z^2 - 6kz + 9k^2)\end{matrix}\right\}$$

$$F_4(z) = \left\{\begin{matrix}\frac{1}{3!k^4}(z^3) \\ \frac{1}{3!k^4}(-3z^3 + 12kz^2 - 12k^2z + 4k^3) \\ \frac{1}{3!k^4}(3z^3 - 24kz^2 + 60k^2z - 44k^3) \\ \frac{1}{3!k^4}(-z^3 + 12kz^2 - 48k^2z + 64k^3)\end{matrix}\right\}$$

Consider $F_4(z)$. Add the polynomials; the sum of the coefficients of each column is zero except in the last column in which it is 4! when the denominators are omitted. Subtract the first polynomial z^3 from the second polynomial and factor out -4, and this becomes $-4(z-k)^3$. From the third polynomial subtract the second and factor out 6, and this becomes $6(z-2k)^3$. Finally subtract from the fourth the third and factor out -4, and this gives $-4(z-3k)^3$. In $F_4(z)$, as written above, it is seen that for $(0, k)$, $F_4(z)=z^3$, (omitting denominators); for $(k, 2k)$, $F_4(z)=z^3-4(z-k)^3$; for $(2k, 3k)$, $F_4(z)=z^3-4(z-k)^3+6(z-2k)^3$; for each new interval another parenthesis is added or subtracted. Hence $F_4(z)$ and also the preceding F's can be written as follows:

$$F_1(z) = \left\{\begin{array}{l} \frac{1}{k}[1 \qquad] \\ \frac{1}{k}[1 - 1] \end{array}\right\} = \sum_{r=0}^{s}(-1)^r\binom{1}{r}(z - rk)^0, \text{ where } s = 0, 1.$$

$$F_2(z) = \left\{\begin{array}{l} \frac{1}{k^2}[z \qquad\qquad\qquad] \\ \frac{1}{k^2}[z - 2(z - k) \qquad] \\ \frac{1}{k^2}[z - 2(z - k) + (z - 2k)] \end{array}\right\} = \frac{1}{k^2}\sum_{r=0}^{s}(-1)^r\binom{2}{r}(z - rk)^1, \quad s = 0, 1, 2.$$

$$F_3(z) = \left\{\begin{array}{l} \frac{1}{2!k^3}[z^2 \qquad\qquad\qquad\qquad] \\ \frac{1}{2!k^3}[z^2 - 3(z - k)^2 \qquad\qquad] \\ \frac{1}{2!k^3}[z^2 - 3(z - k)^2 + 3(z - 2k)^2 \qquad] \\ \frac{1}{2!k^3}[z^2 - 3(z - k)^2 + 3(z - 2k)^2 - (z - 3k)^2] \end{array}\right\}$$

$$= \frac{1}{2!k^3}\sum_{r=0}^{s}(-1)^r\binom{3}{r}(z - rk)^2, \quad s = 0, 1, 2, 3.$$

$$F_4(z) = \begin{cases} \dfrac{1}{3!k^4}[z^3 &] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 &] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3 &] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3 - 4(z-3k)^3 &] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3 - 4(z-3k)^3 + (z-4k)^3 &] \end{cases}$$

$$= \frac{1}{3!k^4} \sum_{r=0}^{s} (-1)^r \binom{4}{r} (z-rk)^3, \quad s = 0, 1, 2, 3, 4.$$

The part of $F_4(z)$ in $(4k, \infty)$ is zero and can be written as

$$\sum_{r=0}^{r=4} (-1)^r \binom{4}{r} (z-rk)^3.$$

In general it can be proven by mathematical iuduction that

$$F_n(z) = \begin{cases} \dfrac{1}{(n-1)!k^n}[z^{n-1} &] \\ \dfrac{1}{(n-1)!k^n}[z^{n-1} - \binom{n}{1}(z-k)^{n-1} &] \\ \dfrac{1}{(n-1)!k^n}[z^{n-1} - \binom{n}{1}(z-k)^{n-1} + \binom{n}{2}(z-2k)^{n-1} &] \\ \cdots\cdots\cdots\

(nk, ∞). It is not difficult to find a law for the coefficients of the polynomials on pages 428, 429. The above may be briefly written as

$$F_n(z) = \begin{cases} F_{n,0}(z) = 0 \text{ for } (-\infty, 0) \\ F_{n,t}(z) = \dfrac{1}{(n-1)!k^n} \displaystyle\sum_{r=0}^{t} (-1)^r \binom{n}{r} (z - rk)^{n-1}, \text{ for } [tk, (t+1)k], \end{cases}$$

where t runs from 0 to n. The interval $[nk, (n+1)k]$ must here mean the interval (nk, ∞).

Consider

$$F_{n,t-1}(z) = \frac{1}{(n-1)!k^n} \sum_{r=0}^{t-1} (-1)^r \binom{n}{r} (z - rk)^{n-1},$$

$$F_{n,t}(z) = \frac{1}{(n-1)!k^n} \sum_{r=0}^{t} (-1)^r \binom{n}{r} (z - rk)^{n-1},$$

$$F_{n,t+1}(z) = \frac{1}{(n-1)!k^n} \sum_{r=0}^{t+1} (-1)^r \binom{n}{r} (z - rk)^{n-1}.$$

It is seen that the first $(n-2)$ derivatives of $F_{n,t-1}(z)$ and $F_{n,t}(z)$ are the same at the point whose abscissa is tk, and also the $(n-2)$ first derivatives of $F_{n,t}(z)$ and $F_{n,t+1}(z)$ are the same at the point whose abscissa is $(t+1)k$. The $(n-1)$th derivatives at these points are discontinuous, which shows that the different curves tend to continue into the adjacent curves completely as continuation of themselves.

Case 2

By Liapounoff's theorem, [2] $F_n(z)$ for large values of n approaches uniformly the Gaussian function $\pi^{-1/2}e^{-z^2}$. By this theorem the probability that the sum of n independent quantities lies between $t_1(2B_n)^{1/2}$ and $t_2(2B_n)^{1/2}$, or that $t_1(2B_n)^{1/2} < z < t_2(2B_n)^{1/2}$, approaches uniformly

$$\pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

provided that

$$\lim_{n \to \infty} \left[(B_n)^{-3/2} \sum_{i=1}^{n} c_i \right] = 0,$$

[2] *Sur l'extension du théoremè limite du calcul des probabilités aux sommes de quantités dépendantes*, by S. Bernstein. Mathematische Annalen, vol. 97 (1926). *Sur une Proposition de la Théorie des Probabilités*, by A. Liapounoff, Bull. de L Acad. Imp. des Sc. de St. Petersbourg, (5), vol. 13 (1900), pages 358–386.

where B_n is the sum of the expected values of the x's raised to the second power and c_i is the expected value of $|x_i|^3$. If the origin is translated to the point $(k/2, 0)$ it is easily seen that the limit is zero and that the probability that $t_1(2B_n)^{1/2} < z < t_2(2B_n)^{1/2}$ approaches

$$\pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

uniformly, or that

$$\lim_{n\to\infty} \int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} F_n(z)dz = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

or that

$$\lim_{n\to\infty} \int_{t_1}^{t_2} (2B_n)^{1/2} \cdot F_n[(2B_n)^{1/2} \cdot z]dz = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt.$$

But since t_1 and t_2 can take on any values it follows that

$$(2B_n)^{1/2} \cdot F_n[(2B_n)^{1/2} \cdot z] \to \pi^{-1/2} e^{-t^2}.$$

Figure 5 shows $\pi^{-1/2}e^{-z^2}$ and $(2B_4)^{1/2} \cdot F_4[(2B_4)^{1/2} \cdot z]$ plotted together. There is less difference between them than the figure indicates, especially at the ends.

By-Products

Since the sum of the polynomials in $F_4(z)$ is $4!$, then by adding them as written on page 430 it follows that (omitting denominators)

$$4\{z^3 - 3(z - k)^3 + 3(z - 2k)^3 - (z - 3k)^3\} = 4!k^3.$$

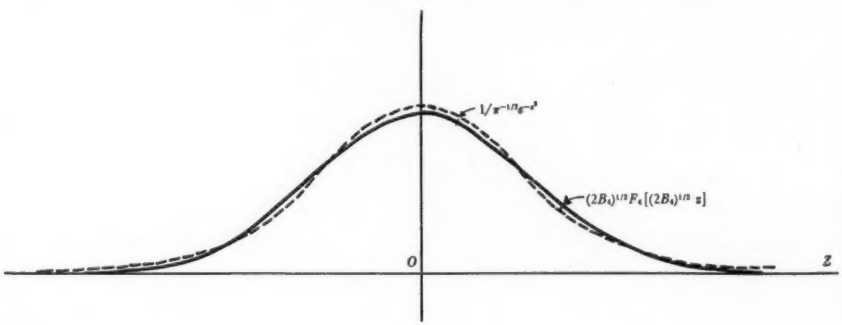

FIG. 5

But since all of the coefficients of the different powers of the z's vanish it follows that

$$(0)^3 - 3(-k)^3 + 3(-2k)^3 - (-3k)^3 = 3!k^3.$$

Let $k=1$ and this becomes

$$(0)^3 - 3(-1)^3 + 3(-2)^3 - (-3)^3 = 3!$$

The coefficients here are the coefficients of the binomial $(a-b)^3$. In general this becomes

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(-rk)^n = n!k^n \text{ and } \sum_{r=0}^{n}(-1)^r\binom{n}{r}(-r)^n = n!$$

Since the constant term in $F_{n,n-1}(z)$ is $(nk)^{n-1}$, the following relation is obtained if the constant term in $F_{n,n-1}(z)$ as written on page 431 is equated to $(nk)^{n-1}$:

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-rk)^{n-1} = (nk)^{n-1}.$$

If $k=1$ and $1/n$, respectively, then

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-r)^{n-1} = n^{n-1},$$

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\left(-\frac{r}{n}\right)^{n-1} = 1.$$

$F_{n,n-1}(z)$ is always equal to $(-z+nk)^{n-1}$. By comparing the coefficients of this with those of $F_{n,n-1}(z)$ on page 431 it is seen that

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-rk)^h = (-1)^{n-1-h}(nk)^h.$$

If $k=1$, then

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-r)^h = n^h(-1)^{n-1-h},$$

where h is an integer and less than or equal to $n-1$ and $n>2$. It will be noted that n^h is an algebraic sum of integers to $n-1$ all raised to the power h with proper coefficients. The coefficients of these integers raised to the power h are the coefficients of the bionomial $(a-b)^n$ with the exception of the last one.

Applications

The mantissas in a five place table of logarithms are rounded-off numbers. For example the mantissa of log 152 is .18184. From what is in the table it is not known whether the 4 came from .1818362, or .1818449, or .18183901, or

.18184001. The 4 is accepted as being the correct fifth digit. The value of the mantissa as written in this table might have been any one of the numbers lying between .181835 and .181845. Any number in this interval as far as is known from the table is equally likely and the probability that the fifth digit came from numbers in this interval is unity and hence the probability that the error lies in the interval $(-.5,+.5)$ is unity and the probability that the error lies outside of this interval is zero. This gives a probability function similar to the ones discussed in this paper. If numbers are multiplied their logarithms are added and it is desirable to know the probability function or error law for the sum of n such functions. $F_n(z)$ developed in this paper gives such a probability function; in fact it gives a probability function for the sum of the errors arising in using any logarithmic table.

A scientist wishes to measure the duration of a certain event and uses a watch which gives the time in seconds. The hand of the watch jumps every second and makes the jump at the moment the particular second is reached. He finds that the duration of a certain event was t seconds. The actual duration might have been $t.001$ seconds or $t.99999$ seconds or any one of the infinite numbers lying between t and $t+1$ seconds. The law of error for the individual event is a constant in the interval $(-.5, .5)$ and zero elsewhere. If the average of n such experiments is obtained the error law for this average is given by a function similar to $F_n(z)$

THE SOLUTION OF CERTAIN DYNAMICAL PROBLEMS

By R. C. COLWELL, Department of Physics, West Virginia University

The type of dynamical problems to be solved in this article is discussed in Routh's *Rigid Dynamics*, (Part 1, page 171) but the method of solution is somewhat different. Routh states that in many dynamical problems the relative motion of the different bodies is all that is required and it is not necessary to find the absolute motion of each body in space. His fundamental theorem is that if any dynamical system is referred to a moving point C, we may reduce C to rest by applying to every element of the system an acceleration equal and opposite to that of C. It is also necessary to suppose that an initial velocity equal and opposite to that of C has been applied to each element. In the method outlined here, it is first assumed that the point C is fixed in space; but no reversed forces are applied to C. After the solution is obtained with C fixed the forces acting are calculated for C in motion and the necessary correction applied directly to the first solution. This method will be exemplified first with the following problem given by Routh. A circular hoop whose weight is nw is free to move on a smooth horizontal plane. It carries on its circumference a small ring weight w, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω about the center of the hoop. Show that the ring

will be at rest on the hoop after a time $(1+n)/\mu\omega$. First suppose that the center of the hoop is fixed in space so that the hoop has a rotation but no translation and is thus compelled to rotate about its center by the friction of the ring w. Solve for the time under these conditions. Let ω_b be the angular velocity of the ring at any time t and ω_h that of the hoop. Then, by equating momenta,

$$I_1\omega = I_1\omega_b + I_2\omega_h; \tag{1}$$

also, from the energies,

$$\tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_b^2 + \tfrac{1}{2}I_2\omega_h^2 + \text{Loss in friction}. \tag{2}$$

Now the centrifugal force acting is $\mu m v_b^2/r$ and the distance passed over along the hoop in time dt is $r(\omega_b-\omega_h)dt$; hence the loss in friction is

$$\int_0^t \mu m r^2 \omega_b^2 (\omega_b - \omega_h)dt. \tag{3}$$

Substituting (3) in (2), we get

$$\tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_b^2 + \tfrac{1}{2}I_2\omega_h^2 + \int_0^t \mu m r^2 \omega_b^2 (\omega_b - \omega_h)dt. \tag{4}$$

Differentiating (4) and using (1), we obtain the differential equation

$$\frac{d\omega_h}{dt} - \frac{\mu m r^2}{I_2}\left\{\frac{I_1\omega - I_2\omega_h}{I_1}\right\}^2 = 0. \tag{5}$$

With the initial conditions $t=0$, $\omega_h=0$, $\omega_b=\omega$ the solution of (5) is

$$\frac{I_2\omega_h}{\omega\{I_1\omega - I_2\omega_h\}} = \frac{\mu m r^2}{I_1}t. \tag{6}$$

At end of time t,

$$I_1\omega = I_1\omega_b + I_2\omega_h. \tag{7}$$

With (7) substituted in (6), the solution becomes

$$t = \frac{I_2}{I_1\mu\omega} = \frac{n}{\mu\omega}. \tag{8}$$

So far the hoop has been regarded as fixed upon a central axis. If, however, it is free to move, the actual acceleration instead of being v^2/r will be given by

$$(nw + w)a' = wv^2/r$$

or

$$a' = \frac{v^2}{[r(n+1)]}. \tag{9}$$

The relative acceleration of the hoop and ring is then

$$a - a' = v^2/r - v^2/[r(n+1)]$$

or

$$a - a' = (v^2/r)\{n/(n+1)\}. \tag{10}$$

The force is weakened in the same ratio and the time lengthened in the inverse ratio.

Hence

$$t = (n/\mu\omega) \times (n+1)/n = (n+1)/\mu\omega. \tag{11}$$

A second problem of the same general type is this: A flat circular disk in two parts (A and B) is lying upon a smooth table. A block m is started in rotation about A with angular velocity ω and coefficient of friction μ(Fig. 1). The part A which is concentric with the part B rotates around B with a frictional couple

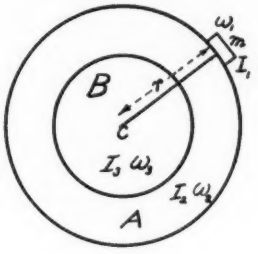

FIG. 1

K. How long before m and A have the same angular velocity? First suppose that the disk is fixed at C so that there is rotation without translation. Then the momental equation gives

$$I_1\omega = I_1\omega_1 + I_2\omega_2 + I_3\omega_3. \tag{12}$$

The loss in friction between m and A is

$$\int_0^t \mu m r^2 \omega_1^2 (\omega_1 - \omega_2)dt.$$

The loss in friction between A and B is

$$\int_0^t K(\omega_2 - \omega_3)dt,$$

where K represents the work done by the frictional couple in turning through

one radian and the friction does not change with the velocity. The energy equation is then

$$(13)\quad \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2 + \int_0^t \mu m r^2 \omega_1^2(\omega_1 - \omega_2)dt + \int_0^t K(\omega_2 - \omega_3)dt.$$

The third equation is

$$(14)\quad K = I_3(d\omega_3/dt).$$

The solution of (12), (13) and (14) for ω_1 gives the differential equation

$$(15)\quad I_1(d\omega_1/dt) + \mu m r^2 \omega_1^2 = 0,$$

from which

$$(16)\quad \omega_1 = I_1\omega/(I_1 + \mu m r^2 \omega t).$$

The three equations also give for ω_2

$$(17)\quad I_2\frac{d\omega_2}{dt} = \frac{\mu m r^2 \omega^2 I_1^2}{(I_1 + \mu m r^2 \omega t)^2} - K.$$

Integrating,

$$(18)\quad I_2\omega_2 = -Kt + I_1\omega - I_1^2\omega/(I_1 + \mu m r^2 \omega t).$$

Equation (14) gives

$$(19)\quad I_3\omega_3 = Kt.$$

With some loss in generality the three differential equations of motion could have been written down at once from the formula $L = I\alpha$:

$$(20)\quad \mu m v^2 = -I_1(d\omega_1/dt),$$

$$(21)\quad \mu m v^2 - K = I_2(d\omega_2/dt),$$

$$(22)\quad K = I_3(d\omega_3/dt).$$

To find the time which elapses before $\omega_1 = \omega_2$, use equations (16) and (18). From these

$$(23)\quad t^2(\mu m r^2 \omega K) + I_1(K - \mu m r^2 \omega^2)t + I_1 I_2 \omega = 0.$$

If as in this case $mr^2 = I_1$ equation (23) reduces to

$$(24)\quad t^2(\mu\omega K) + (K - \mu m r^2 \omega^2)t + I_2\omega = 0.$$

From this equation the time t may be found for any given case when the center C is fixed. If, however, the center is not fixed and the whole disk M slides over a

perfectly smooth surface, it is only necessary as before to write $mM/(M+m)$ for m in equation (24). Thus the transition from fixed to moving bodies becomes extremely simple.

It is easy to show that fixing the center of mass of the whole system gives another simple and complete solution of this type of problem. The reason for this is that when a mass m moves around the circumference of a circle of mass M and radius a we may regard the system as a rigid body, because m and M must always remain the same distance apart. Hence if no outer forces are acting upon the system, the center of mass of the complete system must move in a straight line. The energy due to the motion of the center of mass may then be subtracted from the total energy and the center of mass regarded as fixed in space. Suppose that the hoop M is lying upon a smooth table and the mass m moves upon the circumference of the hoop without friction. Initially the mass m is given a

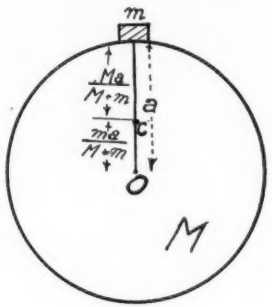

FIG. 2

velocity ω about the center of the hoop; let us find the subsequent motion. The total kinetic energy at once divides into three parts; the energy of m about the center of gravity C, the energy of M about C, and the kinetic energy of both masses regarded as located at the center of gravity. Since $(m+M)V'=mv$, where V' equals the linear velocity of the center of mass, we have

$$\frac{1}{2}m\omega^2a^2 = \frac{1}{2}m\left\{\frac{Ma}{M+m}\right\}^2\omega^2 + \frac{1}{2}M\left\{\frac{ma}{M+m}\right\}^2\omega^2 + \frac{1}{2}\frac{(ma\omega)^2}{M+m}.$$

Thus the center of gravity of the system moves in a straight line parallel to the direction of the original impulse with a velocity $mv/(M+m)$ while the mass m and the center of the circle rotate about it with uniform angular velocity ω.

If, however, there is a coefficient of friction μ between the block and the hoop, the block and hoop lose angular velocity about the center of gravity, but the hoop gains angular velocity about its own center. Let I_{bc} be the moment of inertia of the block about the center of mass of the system, I_{hc} the moment of inertia of the hoop about the same point, I_h the moment of inertia of the hoop

about its own center, ω_b and ω_h the angular velocities of the block and hoop respectively. Then regarding the center of mass C as fixed and omitting its energy from our equations we have

$$I\omega - I\omega_b = I_h\omega_h. \tag{25}$$

In this equation $I = I_{bc} + I_{hc}$ since the two masses M and m move as a rigid body.

The centrifugal force of the rotating block is

$$F = mMa\omega_b^2/(M + m),$$

and the frictional force is

$$\mu mMa\omega_b^2/(M + m);$$

the frictional couple about the center of the hoop is

$$\mu mMa^2\omega_b^2/(M + m) = K'\omega_b^2,$$

where K' is written for the constant $\mu mMa^2/(M+m)$.

The energy equation so far as rotation is concerned is

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_b^2 = \tfrac{1}{2}I_h\omega_h^2 + \int_0^t K'\omega_b^2(\omega_b - \omega_h)dt. \tag{26}$$

The last term in (26) is the energy lost in friction.

From these two equations, we obtain

$$I_h(d\omega_h/dt) = K'\omega_b^2 \text{ and } I(d\omega_b/dt) = -K'\omega_b^2. \tag{27}$$

Initially $t = 0$, $\omega_b = \omega$, and finally

$$\omega_h = \omega_b. \tag{28}$$

Using (27) and (28),

$$\omega_b = \frac{I\omega}{K'\omega t + I}, \qquad \omega_h = \frac{I\omega}{I_h} - \frac{I^2\omega}{I_h(K'\omega t + I)}. \tag{29}$$

Equating the two parts of (29), $t = I_h/K'\omega$, which is the time required for the block to cease sliding along the circumference of the hoop. Putting in the value of K'

$$t = (M + m)/um\omega. \tag{30}$$

In this method of solution it is possible to keep track of the whole energy at any time t. Let us calculate the energies at the time when the block just ceases to slide along the circumference of the hoop, that is when $t = I_h/K'\omega$. First the kinetic energy of the center of gravity due to its linear velocity is

$$m^2a^2\omega^2/2(M + m). \tag{31}$$

The kinetic energy of the hoop about the center of gravity of the system is

$$I_{hc}\omega_b^2/2. \tag{32}$$

The kinetic energy of the block about the center of gravity of the system is

$$I_{bc}\omega_b^2/2. \tag{33}$$

The kinetic energy of the hoop about its own center is

$$\frac{1}{2}Ma^2\omega_b^2 = \frac{1}{2}Ma^2\left\{\frac{I^2\omega^2}{(I+I_h)^2}\right\}. \tag{34}$$

The energy lost in friction is

$$\int_0^t K'\omega_b^2(\omega_b - \omega_h)dt = \frac{K'I^3\omega^3}{I_h}\int_0^t \frac{I_h - K'\omega t}{(I+K'\omega t)^3}dt \tag{35}$$

$$= \frac{1}{2}\frac{II_h}{I+I_h}\omega^2.$$

The sum of (32), (33), (34) and (35) gives the value $I\omega^2/2$ which is the amount of energy originally going into the rotation. Adding to this the linear kinetic energy (31), we obtain the complete energy $ma^2\omega^2/2$.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REPLY TO QUESTION NO. 58[1]

By WM. FITCH CHENEY, JR., Connecticut Agricultural College

QUESTION No. 58[1]: Does the exponential with general base occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, and if so, is Professor Ransom's suggested symbol[2] likely to meet with sufficient approval to be generally adopted?

Some time ago I had occasion to prepare and type a paper involving a great many exponents, with various bases, and I adopted for my own convenience the use of the dollar sign, placed between the base and the exponent. It is

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[1] This Monthly, vol. 37 (1930), p. 188.

[2] This Monthly, vol. 37 (1930), pp. 187–188.

easy to remember, in that anyone will agree that the dollar sign denotes power, and hence such a complicated expression as

$$A^{x^3} - B^{2x-3} + \sqrt{y}$$

may be written all on one line as

$$A\$x\$2 - B\$(2x - 3) + y\$\tfrac{1}{2},$$

which could not be confused with any existing notation. A judicious use of parentheses is the one essential. This has an advantage over the notation proposed by Professor Ransom, in that but one symbol is ordinarily necessary, while his suggested symbol, ⨯ requires two strokes with a back-space between them.

In my opinion, the exponential with general base does not occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, but in any paper which does involve its frequent use, I advocate the use of some special notation, specially defined in that paper.

THE ARC OF THE ELLIPSE

By R. GOORMAGHTIGH, Bruges, Belgium

In the April, 1930 issue[1] of this Monthly, R. A. Johnson has given the following expression for the approximate length S of the ellipse whose semi-axes are a; b:

$$S_0 = \frac{\pi}{2}\left[a + b + 2\left(\frac{a^2 + b^2}{2}\right)^{1/2}\right].$$

Expanding in terms of a and the eccentricity e, we have

$$S_0 = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{180}{2^{14}}e^8 \cdots\right],$$

whereas

$$S = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{175}{2^{14}}e^8 - \frac{441}{2^{16}}e^{10} - \frac{4851}{2^{20}}e^{12} \cdots\right].$$

Up to the *sixth* power of e, the two expansions are the same.

Another remarkable and probably new approximation is

$$S_1 = \frac{\pi}{8}\left[9(a + b) - 5(ab)^{1/2} + 3\left(\frac{a^2 + b^2}{2}\right)^{1/2}\right].$$

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[1] Vol. 37 (1930), p. 188.

The expansion is the same as for S, up to the *tenth* power of e:

$$S_1 = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{175}{2^{14}}e^8 - \frac{441}{2^{16}}e^{10} - \frac{4844}{2^{20}}e^{12} \cdots\right].$$

A Note by the Editor

It may be of interest to emphasize the degree of accuracy of the preceding formula by applying it to an ellipse having the general size and shape of the earth's orbit (which in Newtonian mechanics would be a true ellipse if one neglects, as we shall do, the perturbations due to heavenly bodies other than the sun). Assuming, as at present it seems proper to, that the age of the earth is less than one hundred billion years, then the total distance the earth has traveled since its creation would differ from the approximate distance given by the preceding formula by about the thickness of the sheet of paper on which this is written.

On Strophoidal Curves

By R. Goormaghtigh, Bruges, Belgium

1. In the December, 1928 issue of this Monthly was published a paper by R. M. Mathews and Otto Dunkel,[1] giving a solution of the following question:

A line drawn through the end F of the loop of a right strophoid or a folium of Descartes cuts the curve again in Q and R such that the segment QR subtends a right angle at the node O. Are these results particular cases of a general result for a certain class of cubic curves?[2]

This question suggests a general problem: To determine algebraic plane curves Γ of the n^{th} order, having chords QR passing through a fixed point F and subtending a right angle at another given point O.

2. Consider the polar curve Γ' of Γ, with respect to a circle C having its center at O; then the orthoptic curve of Γ' is a straight line Δ, the polar of F. If Γ'' is the polar curve of Γ' with respect to a circle C' having its center O' on Δ, the chords of Γ'' subtending a right angle at O' are parallel.

Consider rectangular axes with the origin at O', and let the y-axis be parallel to these chords. The equation of the curve Γ'' may be written:

$$\sum_{i=0}^{i=k} A_{k-i}(x)\,y^i = 0, \qquad k \leqq n$$

where $A_{k-i}(x)$ is a polynomial in x of degree not exceeding $n-i$.

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[1] Vol. 35 (1928), pp. 544–547.

[2] This question was proposed as Problem 2928 in this Monthly for the year 1921. (Vol. 28, p. 466.)

Since the degree of Γ' is n there must be at least one of these polynomials with the maximum degree. If we set in turn $y=tx$ and $y=-t^{-1}x$, we shall obtain two equations,

$$\sum_{i=0}^{i=k} A_{k-i}(x)x^i t^i = 0, \qquad \sum_{i=0}^{i=k} A_i(x)x^{k-i}(-1)^{k-i}t^i = 0,$$

which must have the same roots for a given x. Hence

$$A_{k-i}(x)x^i = c(-1)^{k-i}A_i(x)x^{k-i},$$

where c is a constant for the given x. Replacing i by $k-i$ we have also

$$A_i(x)x^{k-i} = c(-1)^i A_{k-i}(x)x^i = c^2(-1)^k A_i(x)x^{k-i}.$$

Since not all of the $A_i(x)$ polynomials can be identically zero, we must have $c^2=1$ and $k=2m$. It will now be convenient to take the coefficient of y^m as $2A_m(x)$.

Thus the equations of the Γ'' curves have the form,

$$\sum_{i=0}^{i=m} A_i(x)[y^{2m-i} + c(-1)^i x^{2m-2i}y^i] = 0, \quad c = +1 \text{ or } c = -1, \quad 2m \leqq n,$$

where the polynomials $A_i(x)$ are quite arbitrary, except that there must be at least one, say $A_i(x)$, of degree $n-2m+i$ if the degree of the curve is n.

The initial remark regarding the relation between Γ and Γ'' enables us to pass from the equation of one to the equation of the other.

Let α, β be the coordinates of θ, r the radius of C, and R the radius of C'. Let X, Y be a point on Γ, and x, y a point on Γ''.

An easy calculation gives the equations of transformation

$$zx = X - \alpha, \quad zy = Y - \beta, \quad R^2 z = \alpha(X-\alpha) + \beta(Y-\beta) + r^2.$$

If we take O as the origin of parallel axes for Γ, we have the simpler equations

$$2x = x', \qquad 2y = y', \qquad R^2 x = \alpha x' + \beta y' + r^2.$$

If we insert these values of x and y in the equation for Γ'', the result, after suppressing the accents, may be written thus:

$$\sum_{i=0}^{i=m} A_i(x, 2)[y^{2m-i} + c(-1)^i x^{2m-2i}y^i] = 0,$$

$$2 = \alpha x + \beta y + r^2, \quad c = +1 \text{ or } c = -1,$$

where $B_i(x, z)$ is an arbitrary homogeneous polynomial in x and z of degree $n-2m+i$; the R^2 of the transformation has been absorbed into the arbitrary constants of $B_i(x, z)$. The coordinates of the point F are $0, -r^2/\beta$, this point being the pole of Δ with respect to C.

We shall now apply these general results to cubics and quartics.

Cubics: Setting $n=3$, $m=1$, we find

$$A_0(x)[y^2 \pm x^2] + A_1(x)[y \mp y] = 0.$$

The only case of interest is then

$$(x+a)(y^2-x^2) + (bx^2+cx+d)y = 0;$$

the cubics have an infinity of chords parallel to $O'y$ and subtending a right angle at O'.

When $d=0$, O' is the double point of the cubic, and the tangents at this point to the two branches of Γ'' are perpendicular. Therefore the tangents at the double point O of the corresponding curve are also perpendicular; since the cubic passes through the point at infinity on $O'y$, F lies on Γ:

If a cubic has a node with perpendicular tangents at this point, the chords subtending a right angle at the node pass through a fixed point lying on the curve.

Quartics: Here $n=4$, $m=1$, 2. Disregarding certain degenerate cases, we find

$$(x^2+ax+b)(y^2-x^2) + (cx^3+dx^2+cx+f)y = 0$$

and

$$y^4+x^4+(ax+b)(y^3-x^2y) + (cx^2+dx+e)y^2 = 0.$$

For instance, when $a=d=e=o$, $c=2$, this last equation represents the trifolium of de Longchamps, and when $a=c=d=e=o$, Cramer's trifolium. The properties of the parallel chords of these curves, subtending a right angle at the node, are well known.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Introduction to the Theory of Fourier's Series and Integrals. Third Edition. By H. S. Carslaw, Macmillan and Co., Limited, London, England, twenty shillings net. Macmillan and Co., New York, seven dollars. 1930.

The first and second editions of this book are too well known to be given space for a review. The newly appearing third edition differs but little from the second. There are certain enlargements amounting to twenty-eight pages in text and exercises, exclusive of appendices. There are also occasional improved wordings. The historical introduction is brought more nearly up-to-date,

and accounts for four pages of the increase. The portion of the book dealing directly with Fourier series contains a few theorems proved with increased generality. Parseval's theorem is a noteworthy addition. Appendix II on Lebesgue integrals is new. This is a readable simple introduction to this subject in English and was much needed. Brief applications to Fourier series are given.

No bibliography is given at the end of the volume in the third as in the second edition. References, however, are given at the ends of chapters.

Persons owning the second edition will probably not find it worth while to buy the third unless wanting it for a class text. Large libraries should have the latest edition of so widely used a book.

TOMLINSON FORT

The History and Significance of Certain Standard Problems in Algebra. By Vera Sanford. Teachers College, Columbia University, 1927. Teachers College Contributions to Education, No. 251. viii+102 pages.

This little book contains a careful and interesting study of the standard verbal problems which appear in textbooks on algebra. These problems the author classifies under three heads; genuine problems, or those arising from real situations; puzzles, or recreations; and an intermediate class called pseudo-real problems in which natural phenomena are combined with unnatural conditions.

There is a discussion of the reasons why problems are taught, the characteristics of good problems, efforts to make problems interesting and real to the student, and some of the ingenious methods formerly used for solving problems.

A chapter is devoted to the history of commercial problems, showing the striking way in which the genuine problems of business arithmetic reflect contemporary conditions and offer material of value in compiling a commercial history of Europe.

Problems from scientific mathematics form another class of genuine problems to which a chapter is devoted. The striking thing about the history of problems of this type is their late appearance. Even so simple an idea as the principle of the lever was overlooked by the makers of problems until the sixteenth century and instances of scientific problems were rare until the time of Newton.

Problems of the recreational type, on the other hand, are for the most part handed down to us from antiquity and have seen little or no development in modern times. Many interesting illustrations of puzzle problems are given both in ancient and in modern dress. Of particular interest are those which are based on progressions; and the author suggests how the teaching of that subject might be improved by the use of such colorful problems instead of the lifeless ones of the usual textbook.

Problems of the pseudo-real type are defended on the ground that they provide a varied material for practice in analysis, and that in many cases

they are more real to the student than problems which actually occur in real situations.

In the closing chapters the author discusses changes in problem material, reasons for the disappearance of certain types of problems, and the survival of others, and finally makes some suggestions as to the lines along which the future development of problems is likely to proceed. In conclusion she says that "in the future puzzle problems will be used because human beings enjoy them; genuine problems will remain because men demand specific and immediate applications of the theory they study; and pseudo-real problems will supplement the other two in providing opportunity for the analytic thinking that forms the most important reason for teaching algebra."

ROSA L. JACKSON

Manual of Trigonometry, for Colleges, Universities, and Technical Schools. By E. C. Kennedy. New York, The Macmillan Co., 1930. 90 pages $8\frac{1}{2}\times 11$ inches. $1.10

This is a collection of 126 problems and exercises in Trigonometry, so arranged as to be available for laboratory work. For the most part there is one exercise to each page, but the last nineteen exercises are listed on one page with no space for solutions. Some attention is paid to the difficulties of solving triangles of extreme forms, and the tables at the end of the book include the common Maclaurin series. There is also a table of the functions of small angles; there are conversion tables for degrees and radians, and tables of the values of constants to fifteen places.

R. A. J.

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3448. *Proposed by Oliver D. Kellogg, Harvard University.*

Prove that whenever the infinite series, with positive terms,

$$u_1 + u_2 + u_3 + \cdots$$

converges, the series

$$\frac{u_1}{r_1}+\frac{u_2}{r_2}+\frac{u_3}{r_3}+\cdots$$

diverges,

$$r_n = u_n + u_{n+1} + u_{n+2} + \cdots$$

being the remainder of the first series after $n-1$ terms.

3449. *Proposed by W. E. Buker, Leetsdale, Pa.*

(1) Find a number X such that X^2+5 and X^2-5 are each square numbers.

Note: This problem was proposed by John of Palermo and solved by Leonardo of Pisa about 1220 A. D. A solution is $X=41/12$; for $(41/12)^2+5=(49/12)^2$; $(41/12)^2-5=(31/12)^2$. How did he arrive at his result?

Reference: Florian Cajori, *History of Mathematics*, pp. 124.

3450. *Proposed by the Late G. B. M. Zerr.*

A starts from Washington towards Baltimore, a distance of 40 miles, and travels at the rate of 10 miles per hour. Before he arrives at Baltimore, a snowstorm starts at Washington and at all places occupying a certain unknown distance towards, but not reaching beyond Baltimore; if A is caught in this storm he must stop until it is over; he is also to receive for this journey a number of dollars inversely proportional to the time occupied in it, at the rate of 50 dollars for one hour. The time when the storm commences is unknown, but all events are equally probable; show that A's expectation is \$11.66.

3451. *Proposed by R. Goormaghtigh, Bruges (Belgium).*

If two conics circumscribed to a triangle are orthogonal at a given point M, their tangents at M being the axes of the conic conjugate to the triangle and having M as center, the product of their normal chords at M is equal to four times the product of their radii of curvature at M.

3452. *Proposed by Charles K. Robbins, Purdue University.*

Solve the functional equation

$$\lambda(x, y)\cdot\mu(x, z) + \nu(x, y) = \omega(y, z).$$

x, y, and z are independent variables and $\partial\omega/\partial y \not\equiv 0$ and $\partial\mu/\partial z \not\equiv 0$.

3453. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given the radical axis of two circles, their circle of similitude, and the length of their line of centers, construct the two circles.

3454. *Proposed by M. S. Knebelman, Princeton Universiy.*

Deduce the explicit expression for $P(m, n)$ from the relation

$$P(m, n) = P(n, m) = \sum_{i=1}^{i=m} P(i, n-1),\ P(1, 1) = 1,$$

where m and n are positive integers.

3455. *Proposed by Solomon Kullback, Brooklyn, N. Y.*

A triangle ABC, with AB constant, is inscribed in a circle. On AC and BC,

construct equilateral triangles, ACD and BCE. Find the locus of the midpoint of ED as C moves along the arc ACB.

SOLUTIONS

3403 [1929, 543]. *Proposed by E. A. Whitman, Carnegie Institute of Technology.*

If in problem 3379, proposed in May, 1929, \$12 and 12 are replaced by "a" dollars and a respectively, for what values of "a" is there a solution?

Problem 3379 reads: Two men own jointly x cows which they sell for x dollars per head, and with the returns buy sheep at \$12 per head. As their income from the cows is not divisible by 12 they purchase a lamb with the remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

Solution by William I. Miller, University of Pittsburgh.

Let the number of sheep, which must be odd, be $2n+1$, and let k represent the price of the lamb in dollars. Then

$$x^2 = a(2n+1) + k,\ 0 < k < a,$$

or

$$x^2 \equiv a + k \pmod{2a},\ 0 < k < a.$$

The problem is to determine a so that the congruence has solutions for just one value of k. By trial, $a=3$, 6, 8, 10, or 12. It is not necessary to try values for a greater than 18, for if $a \geqq 19$, there will always be at least two perfect squares between a and $2a$, so that k cannot be unique.

To prove the last statement, suppose we can choose n so that $2a > (n+1)^2$ where $n^2 > a \geqq (n-1)^2$. This will be possible if $2(n-1)^2 > (n+1)^2$, or if $n \geqq 6$, and hence if $a \geqq 25$. The statement is obviously true if $19 \leqq a \leqq 24$.

Also solved by W. E. Buker, and the Proposer.

Note by the Editors: If a is odd it may be shown that k is not unique for $a>3$. For, set $x=ma+i$, where m is an odd positive integer and $i=1$ or 2. Then $x^2=a[am^2+2mi]+i^2$ and the brackets contain an odd integer. Hence k may be taken as 1 or 4. This reduces the trials above to $a=3$ and to the even integers.

3406 [1930, 37]. *Proposed by William P. Parker, Pyengyang, Chosen, Korea.*

Find the condition that $(\alpha x+\beta y+\gamma z)\ (\alpha_1 x+\beta_1 y+\gamma_1 z)-(\alpha_2 x+\beta_2 y+\gamma_2 z)^2$ may be resolved into two linear factors.

Solution by E. D. Rainville, University of Colorado.

The expression under consideration is a ternary quadratic form. By a well-known theorem, (Maxime Bôcher, *Introduction to Higher Algebra*, page 137), the necessary and sufficient condition that this expression be reducible is that

$$\Delta = \begin{vmatrix} 2(\alpha\alpha_1 - \alpha_2^2) & \alpha\beta_1 + \beta\alpha_1 - 2\alpha_2\beta_2 & \alpha\gamma_1 + \gamma\alpha_1 - 2\alpha_2\gamma_2 \\ \alpha\beta_1 + \beta\alpha_1 - 2\alpha_2\beta_2 & 2(\beta\beta_1 - \beta_2^2) & \beta\gamma_1 + \gamma\beta_1 - 2\beta_2\gamma_2 \\ \alpha\gamma_1 + \gamma\alpha_1 - 2\alpha_2\gamma_2 & \beta\gamma_1 + \gamma\beta_1 - 2\beta_2\gamma_2 & 2(\gamma\gamma_1 - \gamma_2^2) \end{vmatrix} = 0.$$

If we expand Δ into 27 third order determinants in the usual way, it is immediately obvious that

$$\Delta = 2\begin{vmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \beta_1 & \beta_2 \\ \gamma & \gamma_1 & \gamma_2 \end{vmatrix}^2 = 0.$$

That is, the necessary and sufficient condition desired is

$$\begin{vmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \beta_1 & \beta_2 \\ \gamma & \gamma_1 & \gamma_2 \end{vmatrix} = 0.$$

Note by Otto Dunkel. Geometric considerations enable us to discover the required condition. If λ, μ, ν denote homogeneous linear expressions in x, y, z, then $\lambda\mu - \nu^2$ set equal to zero gives a conic for which $\nu = 0$ is the chord of contact of the two tangents $\lambda = 0$, $\mu = 0$. If the conic degenerates into two straight lines, then these two straight lines pass through a point common to the three lines $\lambda = 0$, $\mu = 0$, $\nu = 0$. The converse may also be considered geometrically.

An analytical proof may be given which depends mainly upon the simple theorems for linear dependence. We shall prove that:

A necessary and sufficient condition that

(1) $$\lambda\mu - \nu^2$$

where λ, μ, ν are homogeneous linear expressions in x, y, z, shall reduce to the product of two linear expressions, is that λ, μ and ν shall be linearly dependent, that is that

(2) $$a\lambda + b\mu + c\nu \equiv 0,$$

where a, b, c are constants not all of which are zero.

If (2) is true, then we may solve for at least one of the three expressions, say λ, in terms of the other two, μ and ν. Inserting this expression for λ in (1), we shall obtain a homogeneous quadratic expression in μ and ν. Such an expression can always be factored, and hence (1) reduces to the product of two linear factors.

Suppose now that

(3) $$\lambda\mu - \nu^2 \equiv \alpha\beta,$$

where α and β are linear in x, y, z. Suppose first that $\alpha \equiv c\nu$, $\beta \equiv c'\nu$, where c and c' are constants. Since $\nu = 0$ for an infinite set of values of x, y, z, either λ or μ must vanish for more than one set of these values. Suppose it is λ, then λ and ν are linearly dependent; consequently, λ, μ, ν are linearly dependent. We may now suppose that not both of the above identities are true, say α is not linearly dependent upon ν. Since λ and α vanish simultaneously for one set of values of x, y, z, this set of values must make ν also vanish. Hence λ, ν, α are linearly dependent. If the coefficient of α in this linear relation is zero, then λ and ν are

linearly dependent, and, therefore, λ, μ, ν are linearly dependent. Suppose now the coefficient of α is not zero, we may then write $a\lambda+b\nu\equiv\alpha$ where a is not zero since this case is excluded here. By a similar argument we arrive at the result $a'u+b'\nu\equiv\alpha$. Hence $a\lambda-a'\mu+(b-b')\nu\equiv 0$, where neither a nor a' is zero. Since this exhausts the possible cases the theorem is proved.

Also solved by W. E. Buker, E. F. Cox, Raymond Garver, J. H. Neelley, A. Pelletier, . J. ORamler.

3407 [1930, 37]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

A ray of light emanating from a fixed source L is reflected by a flat mirror at a point M so that the reflected ray passes through a given point I. Find the locus of M when the mirror revolves about a fixed axis s.

Solution by A. Pelletier, Montreal, Canada.

Let s be the z-axis, and let the coordinates of I and L be $(a, 0, 0)$ and (b, c, d), respectively. For any position of the mirror, LM passes through I_1, the point symmetric to I with regard to the mirror in that position. Thus the locus of I_1, is a circle of radius $OI=a$ which lies in the xy-plane. Hence the point M lies on a cone with the vertex L and with this circle as a base. The equation of this cone is

$$(dx - bz)^2 + (dy - cz)^2 = a^2(d - z)^2. \tag{1}$$

Next let us consider the projection of the path of M on the xy-plane. Let ON be the trace of the mirror on the xy-plane, and let α be the angle which ON makes with the x-axis. Then the coordinates of I_1 are $a\cos 2\alpha$, $a\sin 2\alpha$; and the equation of the projection of this path is obtained by eliminating α from

$$y = x\tan\alpha,\ (y - c)/(x - b) = (c - a\sin 2\alpha)/(b - a\cos 2\alpha). \tag{2}$$

The elimination of α gives

$$(by - cx)(x^2 + y^2) + a(c - y)(x^2 - y^2) + 2a(x - b)xy = 0. \tag{3}$$

The equations (1) and (3) determine the locus of M in space. The points L and I are the extremities of the actual path of M.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Harvard University has conferred an honorary doctorate on K. T. Compton, the newly elected president of the Massachusetts Institute of Technology.

Indiana University has conferred honorary doctorates on Dr. C. O. Lampland and Dr. V. M. Slipher of the Lowell Observatory, Flagstaff. .

Lake Forest College has conferred an honorary doctorate on Professor W. D. MacMillan, of the University of Chicago.

Professors P. Franklin and D. J. Struick, of the Massachusetts Institute of Technology, have been elected Fellows of the American Academy of Arts and Sciences.

A portrait of Professor E. H. Moore, of the University of Chicago, painted by Mr. Ralph Clarkson, has been presented to that University by Professor Moore's former students. It will be hung in the new Eckhart Hall of Mathematics, Physics, and Astronomy.

Dr. W. F. Willcox, of Cornell University, represented the United States at the meeting of the International Institute of Statistics which was held at Tokyo in September, 1930.

Dr. V. W. Adkisson has been appointed assistant professor of mathematics at the University of Arkansas.

Mr. J. G. Adshead has been appointed assistant professor of mathematics at Dalhousie University.

Assistant Professor J. P. Ballantine, of the University of Washington, has been promoted to an associate professorship of mathematics.

Dr. P. M. Batchelder has been promoted to an associate professorship of mathematics at the University of Texas.

Leon H. Bunyan, who has been an instructor at the University of Wisconsin, Extension Division, Milwaukee, Wisconsin, has been appointed assistant professor in the department of mathematics of Rutgers University, New Brunswick, N. J.

Associate Professor C. W. Cobb, of Amherst College, has been promoted to a professorship of mathematics.

Dr. H. P. Evans has been promoted to an assistant professorship of mathematics at the University of Wisconsin.

Professor J. W. Glover, of the University of Michigan, has been appointed president of the Teachers Insurance and Annuity Association of America.

Assistant Professor Cornelius Gouwens, of Iowa State College, has been promoted to an associate professorship of mathematics.

Dr. Lois W. Griffiths, of Northwestern University, has been promoted to an assistant professorship of mathematics.

Assistant Professor Einar Hille, of Princeton University, has been promoted to an associate professorship of mathematics.

Dr. J. J. L. Hinrichsen has been appointed assistant professor of mathematics at Iowa State College.

Assistant Professor Clyde M. Huber, of Rutgers University, has been appointed professor and head of the department of mathematics at Atlantic University, Virginia Beach, Va.

Dr. B. W. Jones has been appointed assistant professor of mathematics at Cornell University.

Dr. A. Marguerite Lehr, of Bryn Mawr College, has been promoted to be an associate in mathematics.

Dr. C. O. Oakley has been promoted to an assistant professorship of mathematics at Brown University.

Associate Professor J. F. Reilly, of the State University of Iowa, has been promoted to a professorship of mathematics.

Assistant Professor C. A. Rupp has been promoted to an associate professorship at Pennsylvania State College.

Professor C. D. Smith, of Louisiana College, has been appointed professor and head of the department of mathematics at the Mississippi Agricultural and Mechanical College.

Assistant Professor W. M. Whyburn has been promoted to an associate professorship of mathematics at the University of California at Los Angeles.

Dr. B. C. Wong has been promoted to an assistant professorship of mathematics at the University of California at Berkeley.

A Correction

There is a misprint in line 7 of J. P. Ballantine's article entitled *A peculiar Function*, in this Monthly, May 1930, p. 250. Instead of

$$r = R \cos \tfrac{1}{2}\theta \quad \text{when} \quad 90 < \theta \leq 180$$

read

$$r = R \sin \tfrac{1}{2}\theta \quad \text{when} \quad 90 < \theta \leq 180.$$

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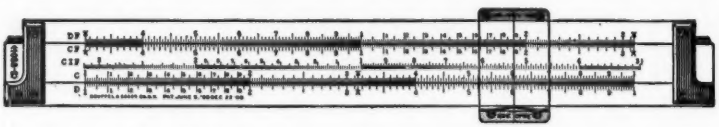

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The Fourth Carus Mathematical Monograph

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the fourth number has been published and is ready for distribution. The title of this Monograph is "Projective Geometry" by Professor JOHN W. YOUNG of Dartmouth College, now President of the Association. The preceding numbers are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ.

The price of these Monographs is $1.25 to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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DIRECTORY

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BOOKS FOR REVIEW should be sent to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

Illinois, Lake Forest, Ill., May 2-3.

Indiana, Earlham College, May 2-3.

Iowa, Ames, Iowa, May 2-3.

Kansas, February 15.

Kentucky, Lexington, Ky., April 5.

Louisiana-Mississippi, Cleveland, Miss., March 7-8.

Maryland-District of Columbia-Virginia, May 10.

Michigan, Ann Arbor, Mich., March 22.

Minnesota, Carleton College, May 17.

Missouri.

Nebraska, Peru, Neb., May 9.

Ohio, Columbus, Ohio, April 3.

Philadelphia, Philadelphia, Pa., November 29.

Rocky Mountain, Denver, Colo., April 11-12.

Southeastern, Atlanta, Ga., May 2-3.

Southern California, University of Southern California, Los Angeles, Calif., March 8.

Texas.

Affiliated Organizations: The New England Association of Teachers of Mathematics. The National Council of Teachers of Mathematics.

THE FOURTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The fourteenth summer meeting of the Mathematical Association of America was held, by invitation, at Brown University, Providence, R.I., on Monday and Tuesday, September 8–9, 1930, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Three hundred five were present at the meetings, including the following one hundred seventy members of the Association.

C. R. ADAMS, Brown University
ETHEL M. ANDERTON, Mount Holyoke College
R. C. ARCHIBALD, Brown University

CLARA L. BACON, Goucher College
W. D. BATEN, University of Michigan
A. A. BENNETT, Brown University
THEODORE BENNETT, University of Wisconsin
E. M. BERRY, Lynchburg College
G. D. BIRKHOFF, Harvard University
H. F. BLICHFELDT, Stanford University
G. A. BLISS, University of Chicago
M. G. BOYCE, Western Reserve University
R. W. BRINK, University of Minnesota
GERTRUDE L. BROCKETT, Brown University
J. A. BULLARD, University of Vermont
L. H. BUNYAN, Rutgers University
W. H. BUSSEY, University of Minnesota
A. D. BUTTERFIELD, University of Vermont

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B. H. CAMP, Wesleyan University
C. C. CAMP, University of Nebraska
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